WAVE PROPAGATION AND IR/UV MIXING IN NONCOMMUTATIVE SPACETIMES[1]

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ABSTRACT

In this thesis I study various aspects of theories in the two most studied examples of noncommutative spacetimes: canonical spacetime ($[x_\mu, x_\nu] = \theta_{\mu\nu}$) and κ-Minkowski spacetime ($[x_i, t] = \kappa^{-1} x_i$). In the first part of the thesis I consider the description of the propagation of "classical" waves in these spacetimes. In the case of κ-Minkowski this description is rather nontrivial, and its phenomenological implications are rather striking. In the second part of the thesis I examine the structure of quantum field theory in noncommutative spacetime, with emphasis on the simple case of the canonical spacetime. I find that the so-called IR/UV mixing can affect significantly the phase structure of a quantum field theory and also forces us upon a certain revision of the strategies used in particle-physics phenomenology to constrain the parameters of a model.

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Introduction

It is widely believed that spacetime at lengths scales of the order of the Planck length ($L_p = \sqrt{\hbar G/c^3} \simeq 10^{-33} cm$) is no longer describable as a smooth manifold. Nonclassical properties of spacetime are expected to affect processes involving particles of ultrahigh energy. There are in principle, at least, two ways to address the issue of nonclassical properties of spacetime and their observable effects. A first way, which is in a sense more fundamental, is the one of trying to construct a whole quantum theory of spacetime according to some picture of the unification of General Relativity with Quantum Mechanics, and then look at its low-energy predictions. This is, for instance, the strategy adopted by the most studied and, presently, most promising approaches to Quantum Gravity, such as string theory and (canonical) loop quantum gravity. A possible reason of concern for these type of approaches is that one has to make a correct guess of the laws of Nature at an energy scale ($E_p = \frac{1}{L_p} \simeq 10^{19} Gev$) that is very far away from the energy scales currently explored in the laboratory ($E_{\exp} \simeq 10^2 Gev$). This is a difficult task as the one of trying to grasp the details of weak interaction ($E_W \simeq 10^2 Gev$) just by studying the properties of the common macroscopic objects of everyday life.

A second, more humble way to approach research on nonclassical properties of spacetime is the one of effective theories. One tries to model some nonclassical spacetime effects without necessarily assuming full knowledge of the short-distance structure of spacetime. Among these proposals there has been strong interest in the idea that it might be fundamentally impossible to fully specify the position of a particle. This can formalized through a spacetime uncertainty principle of the form

$$\Delta x_\mu \Delta x_\nu \geq \theta_{\mu\nu}, \tag{1}$$

that would introduce in spacetime an uncertainty relation which is analogous to the Heisenberg phase-space uncertainty relation $\Delta x \Delta p > \hbar/2$. At the formal level the Heisenberg uncertainty principle turns out to be described by Hermitian operators satisfying the noncommutativity relation $[x, p] = i\hbar$. Following the analogy one is led to consider similar commutation relations in the spacetime sector

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}(x), \tag{2}$$

which imply noncommutativity of spacetime.

In the last few years noncommutative spacetimes have attracted interest from many authors not only for the reasons we have outlined, which make them interesting on their own, but also since they emerge as a possible description of spacetime in theory constructed without assuming, a priori, noncommutativity of spacetime. In particular the so-called canonical spacetimes[2]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}, \tag{3}$$

emerge in the descriptions of string theory in presence of external fields [1, 2], and it is also a useful-alternative tool in the description of electronic systems in external-magnetic field [3, 4, 5]. While, certain Lie-algebra noncommutative spacetime ($[x_\mu, x_\nu] = iC^\alpha_{\mu\nu}x_\alpha$), emerge in the framework of some approaches to Quantum Gravity that predict a minimum wavelength. This is, for instance, the case of the so-called κ-Minkowski spacetime

$$\begin{aligned} [x_i, x_0] &= i\frac{1}{\kappa}x_i, \\ [x_i, x_j] &= 0, \end{aligned} \tag{4}$$

that is connected by duality relation to a deformation of the Poincaré group as a quantum group known as κ-Poincaré. This κ-Poincaré quantum group has been extensively studied in literature [6, 7, 8, 9, 10, 11] especially since it involves not only an invariant velocity scale c, but also an invariant length scale $\lambda = \frac{1}{\kappa} \simeq L_p$.

In this thesis we will analyze the popular noncommutative spacetimes of Eqs.(3) and (4), focusing on some key theoretical issues and their phenomenological implications. We will start our study from the problem of wave construction and propagation in these types of noncommutative spacetime. The analysis of waves is a key element for planned experimental studies which hope to detect nonclassical effects of spacetime. It is expected in fact [12, 13, 14, 15, 16] that spacetime noncommutativity might manifest with detectable modifications of the usual laws of particle production and propagation. Moreover, over the last few years, there has been a sharp increase in the interest toward experimental investigations of Planck-scale effects (see, *e.g.*, Refs. [17, 18, 19, 20, 21, 22, 23, 24]). In particular, studies such as the ones planned for the GLAST space telescope [25] would be sensitive to small, Planck-scale suppressed, deviations from the special-relativistic relation between group velocity and momentum. Within the framework of Planck-scale spacetime noncommutativity such modifications of the relation between group velocity and momentum are often encountered. In this respect, while canonical spacetime is expected to play a minor role (alterations of in-vacuum propagation seem to be relevant mainly for polarization-connected effects [26]), such effects are expected to have deeper implications in κ-Minkowski spacetime, where they are believed to manifest in the deformation of in-vacuum dispersion relations and (related) modifications of the energy thresholds for certain

[2] Here $\theta_{\mu\nu}$ is a coordinate-independent antisymmetric matrix.

particle-production processes [21, 22]. The effect of spacetime discreteness should be significant for particles of high energy. Deviations from standard propagation are expected to be suppressed as powers of the ratio between Planck length and particle wavelength, but in spite of this huge suppression the mentioned planned experiment should be able to see them. In preparation for these planned experimental studies, on the theory side there has been an intense debate on the proper description of the notion of velocity in a noncommutative spacetime. Here (in Chapter 2) we analyze velocity in κ-Minkowski spacetime. This notion of velocity had been already discussed in several studies (see Refs. [7, 10] and references therein), under the working assumption that the relation $v = dE(p)/dp$, which holds in Galilei spacetime and Minkowski spacetime, would also hold in κ-Minkowski. This leads to interesting predictions as a result of the fact that, upon identification of the noncommutativity scale $1/\kappa$ with the Planck length L_p, the dispersion relation $E(p)$ that holds in κ-Minkowski is characterized by Planck-length-suppressed deviations from its conventional commutative counterpart. Recently the validity of $v = dE(p)/dp$ in κ-Minkowski has been questioned in the studies reported in Refs. [27, 28]; moreover in the study reported in Ref. [11] the relation $v = dE(p)/dp$ was considered on the same footing as some alternative relations. Especially in light of the plans for experimental studies, this technical issue appears to be rather significant. We approach the study of κ-Minkowski adopting the line of analysis proposed in Ref. [10]. We argue that key ingredients for the correct derivation of the relation between group velocity and momentum are: a fully developed κ-Minkowski differential calculus, and a proper description of energy-momentum in terms of generators of translations. Our analysis provides support for the adoption of the formula $v = dE(p)/dp$, already assumed in most of the κ-Minkowski literature. We discuss the *ad hoc* assumptions which led to alternatives to $v = dE(p)/dp$ in Refs. [27, 28], and we find that the analysis in Ref. [28] was based on erroneous implementation of the κ-Minkowski differential calculus, while the analysis in Ref. [27] interpreted as momenta some quantities which cannot be properly described in terms of translation generators.

We also discuss the proposals of construction of field theories on noncommutative spacetimes based on the Weyl-Moyal map. We focus mainly on the canonical spacetime which up to now has been the most extensively studied. A key characteristic of field theories on canonical spacetimes, which originates from the commutation relation, is nonlocality. At least in the case of space/space noncommutativity ($\theta_{0i} = 0$), to which we limit our analysis for simplicity[3], this nonlocality is still tractable although it induces an intriguing mixing of the ultraviolet and infrared sectors of the theory. This IR/UV mixing has wide implications both for the phenomenology and for the theoretical understanding of these models. One of the manifestations

[3]The case of space/time noncommutativity ($\theta_{0i} \neq 0$) is not necessarily void of interest [29, 30, 31, 32, 33], but it is more delicate, especially in light of possible concerns for unitarity. Since our analysis is not focusing on this point we will simply assume that $\theta_{0i} = 0$.

of the IR/UV mixing is the emergence of infrared (zero-momentum) poles in the one-loop two-point functions. In particular one finds a quadratic pole for integer-spin particles in non-SUSY theories [34], while in SUSY theories the poles, if at all present, are logarithmic [35, 36, 37]. It is noteworthy that these infrared singularities are introduced by loop corrections and originate from the ultraviolet part of the loop integration: at tree level the two-point functions are unmodified, but loop corrections involve the interaction vertices, which are modified already at tree level.

There has been considerable work attempting to set limits on the noncommutativity parameters θ by exploiting the modifications of the dressed/full propagators [26, 38] and, even more, the modifications of the interaction vertices [39, 40, 41]. Most of these analyses rely on our readily available low-energy data. The comparison between theoretical predictions and experimental data is usually done using a standard strategy (the methods of analysis which have served us well in the study of conventional theories in commutative spacetime). We are mainly interested in understanding whether one should take into account some of the implications of the IR/UV mixing also at the level of comparing theoretical predictions and data. It appears plausible that the way in which low-energy data are used to constrain the noncommutativity parameters may be affected by the IR/UV mixing. These limits on the entries of the θ matrix might not have the usual interpretation: they could be seen only as "conditional limits", conditioned by the assumption that no contributions relevant for the analysis are induced by the ultraviolet. The study we report here is relevant for this delicate issue. By analyzing a simple noncommutative Wess-Zumino-type model, with soft supersymmetry breaking, we explore the implications of ultraviolet supersymmetry on low-energy phenomenology. Based on this analysis, and on the intuition it provides about other possible features of ultraviolet physics, we provide a characterization of low-energy limits on the noncommutativity parameters.

To explore the consequences of the IR/UV mixing using a nonperturbative technique (effectively resumming infinite series of 1PI Feynman diagrams) we present an application of the Cornwall-Jackiw-Tomboulis (CJT) [42] formalism to the noncommutative scalar theory. The CJT formalism has proven to be a powerful nonperturbative approach to the problem of phase transition in QFT in commutative spacetime (see Refs. [42, 43]) and in Thermal-Quantum-Field Theories [43, 44, 45]. These theories suffer from severe infrared problems which recall those related to the IR/UV mixing in noncommutative theories. We analyze in the CJT formalism the issues of phase transitions and renormalizability of (canonical) noncommutative scalar-φ^4 theory. We discuss the applicability of the CJT formalism in a noncommutative framework. Then we focus on the so-called Hartree approximation that is equivalent to a selective resummation of the diagrams of the common perturbative expansion, in particular summing all the so-called "daisy" and "super daisy" diagrams. In the Hartree approximation, and under the hypothesis of translational invariance of the vacuum, we calculate the effective potential that

will be expressed in terms of a mass parameter to be determined as solution of a gap equation. In particular, we analyze the renormalizability of the gap equation and of the potential. We find that whereas in the symmetric phase (characterized by a zero vacuum-expectation value of the scalar field) both the gap equation and the potential are renormalizable, in the broken-symmetry phase (characterized by a non-zero vacuum-expectation value of the scalar field) the gap equation and the potential do not renormalize. These results appear to reinforce the hypothesis that in noncommutative theories because of the IR/UV mixing there might be a stable (or quasi-stable) translation-non-invariant vacuum.

This thesis is structured as follows. In Chapter 1, that is a review chapter, we introduce spacetime noncommutativity focusing on canonical noncommutativity and κ-Minkowski noncommutativity. We discuss the physical frameworks from which they emerge and study their symmetries, and we also briefly review the (rather technical) mathematical structures that are used in the rest of the thesis.

In Chapter 2, that is based on Ref. [46], we analyze wave propagation both in canonical and κ-Minkowski spacetimes. We focus on the concept of group velocity in view of its relevance for planned experimental studies and stress the differences between the two noncommutative spaces.

In Chapter 3, that is a review chapter, we introduce the quantum field theories in noncommutative spacetime. We outline the standard strategy of quantization, based on the Weyl-Moyal map, and discuss its application to the noncommutative spacetimes. While this procedure of quantization can be successful implemented in canonical-noncommutative spacetime, in κ-Minkowski spacetime it is still not well understood. Then, for the rest of the chapter (and of the thesis) we focus only on canonical spacetime and especially on the problems related to the IR/UV mixing.

In Chapter 4, that is based on Ref. [47], we study the implications of the IR/UV mixing for the phenomenology. We discuss how the IR/UV mixing renders the low-energy prediction sensitive to the (unknown) structure of the UV sector. We illustrate our point in a Wess-Zumino model with soft supersymmetry breaking. We show how ultraviolet supersymmetry can modify drastically the low-energy predictions of the theory, and analyze the implications of these observations for studies attempting to constrain the non-commutativity parameters using low-energy data. We conclude on the implication of this analysis for the selection of reliable theoretical models from low-energy-experimental data.

In Chapter 5, that is based on Ref. [48], we review the CJT formalism and apply it to the noncommutative scalar theory. We analyze the effective potential and discuss the problem of its renormalization both in the symmetric phase and in the uniformly-broken phase. We focus on the so-called "bubble approximation". We study under the hypothesis of translational invariance of

the vacuum, the gap equation and the effective potential discussing also the planar limit (strong noncommutativity) and the commutative limit. Then we discuss the renormalization of both the gap equation and the potential in the symmetric and in the broken-symmetry phase. In Chapter 6 we present our conclusions.

Chapter 1

Noncommutativity in physics

* The idea of extending to the spacetime sector noncommutativity of the phase space is rather old. The first paper on this subject was published by Snyder in 1947 [49] although the first proposal of noncommutativity among coordinates is attributed to Heisenberg in the late 1930s. Heisenberg hoped that noncommutativity would improve the short-distance singularities typical of the quantum field theories while extending the uncertainty relations to the coordinate sector. Apparently [4] Heisenberg suggested this idea to Peierles who used it in a phenomenological approach to the study of electronic systems in an external field [3]. Then Peierles told about Heisenberg's idea to Pauli, who in turn told to Oppenheimer, who told it to his student Snyder.

In this chapter we briefly review the main arguments that, beyond Heisenberg's idea, support the introduction of noncommutativity in the spacetime. Then we will focus on canonical spacetime and κ-Minkowski spacetime.

The interest for the canonical spacetime is motivated by the fact that, since it involves coordinate-independent commutators, it is the simplest noncommutative spacetime and can also be seen as the zeroth-order approximation of a very general class of noncommutative spacetimes. We briefly discuss how canonical noncommutativity plays a role in the description of electronic systems in strong magnetic field and how it emerges in certain string theories in strong external field.

The interest in κ-Minkowski spacetime comes from the fact that it is dual to a quantum deformation of the Poincaré group which has recently attracted a lot of interest especially for the relevance in Double Special Relativity theories. We discuss the basic features of such Double Special Relativity (DSR) theories. We analyze how DSR theories find precise realization in the mathematical scheme of κ-deformed (quantum) Poincaré group and we discuss the duality relation that leads to κ-Minkowski spacetime. Then we also discuss how κ-Minkowski spacetime and canonical spacetime emerge as the only possible flat noncommutative spacetimes that can be constructed from a general Lie-algebra spacetime and from the hypothesis of invariance under

[0]* In this Chapter we review the issue of noncommutativity in physics.

undeformed spatial rotations (κ-Minkowski spacetime) or undeformed translations (canonical spacetime).

1.1 General arguments for noncommutativity in spacetime

As mentioned the first studies of spacetime noncommutativity were motivated by the analogy with quantum mechanics and by the idea that one could use noncommutativity of spacetime coordinates to introduce an effective cutoff to cure the divergences appearing in quantum field theories. We will analyze the issue of the regularization of the divergences in the next chapters when a formulation of the quantum field theory will have been introduced. Here we consider the analogy with quantum mechanics.

Starting from the classical phase space in which a point can be localized without any limit of precision, one can define the quantum phase space by substituting canonical positions and momenta x_i and p_i with Hermitian operators which obey to the Heisenberg commutation relations $[x_i, p_j] = i\hbar\delta_{ij}$. From the uncertainty relations

$$\Delta x_i \Delta p_j \geq \frac{1}{2}\hbar\delta_{ij} \tag{1.1}$$

follows that in a quantized phase-space there exists a maximum in the precision with which a point can be localized. Upon quantization the classical phase-space becomes smeared out and the notion of point must be replaced by that of Planck cell whose characteristic size is $O(\hbar)$. One expects to become sensitive to the quantization of the phase space when the action of the system under consideration, which has the same dimensions of $\hbar$, is of the size of the Planck constant[1].

By analogy with the phase-space quantization one can attempt to represent spacetime by replacing classical coordinates x_i with the Hermitian generators of a noncommutative algebra of functions satisfying the commutation relation

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}. \tag{1.2}$$

Since coordinates do not commutate they cannot be simultaneously diagonalized and one can expect spacetime uncertainty relations of the form

$$\Delta x_i \Delta x_j \geq \frac{1}{2}|\theta_{ij}| \tag{1.3}$$

that, as already emphasized in the Introduction, may be appropriate for a description of quantum spacetime fluctuations.

In this case the notion of spacetime point loses its meaning. Spacetime points are replaced by cells of area of size $|\theta_{ij}|$. The quantum fluctuations of the spacetime prevent the exact localization

[1]In the path integral language one obtains sensitivity to the quantum phase-space when $S/\hbar \sim 1$ and also the paths different from the classical one ($\delta_x S = 0$) give contributions to the dynamics.

of the events inside this area. Now we could expect to acquire sensitivity to the quantum structure of spacetime when the size of the system, or of the probe, is of order of $\sqrt{|\theta_{ij}|}$[2].

Expectations similar to (1.3) for the description of spacetime at extremely small distances come also from string theory presently one of the best candidates for a quantum theory of gravity. Strings possess an intrinsic length-scale l_s, and using string states as a probe it is not possible to observe distances smaller than l_s. It is not surprising that in certain string theories [50, 51, 52] (but also in many other approaches to quantum gravity, see [53, 54]) modified Heisenberg uncertainty relation has been found of the form

$$\Delta x \gtrsim \frac{\hbar}{2}\left(\frac{1}{\Delta p} + l_s^2 \Delta p\right) \tag{1.4}$$

If one minimizes (1.4) with respect to Δp one obtains a lower bound on the measurability of lengths in spacetime: $\Delta x \simeq l_s$. Therefore also from the point of view of string theory the space of the configurations is smeared out and the notion of point is meaningless. More generally uncertainty relation has been postulated [55, 56] of the form

$$\Delta x_i \Delta x_j \geq l_s^2 \tag{1.5}$$

that directly follows from noncommutativity of spacetime (1.2).

1.2 Examples of noncommutative canonical spacetimes

Beyond these general arguments there are explicit examples in physics in which noncommutativity plays an important role and some theoretical frameworks in which noncommutativity is not assumed a priori, but eventually follows from the analysis. In this section we want to discuss some of these examples showing how canonical noncommutativity of coordinates arises.

1.2.1 Canonical noncommutativity in condensed matter systems

A first physical system which exhibits noncommutativity comes from condensed matter (see, e.g., [57]). Let us consider a point-particle moving on a 2-d plane[3] in presence of an external magnetic field $\vec{B}$ perpendicular to the plane. The equation of motion for the particle is

$$m\ \dot{v}^i = \frac{e}{c}\varepsilon^{ij} v^j |\vec{B}| + f^i(\vec{r}) \tag{1.6}$$

where $\vec{r} \equiv (x, y)$ gives the position of the particle, $\vec{v}$ is the particle velocity and $\vec{f}$ are some external conservative forces: $\vec{f} = -\vec{\bigtriangledown} V$.

[2]The nature of the uncertainty principle implied by (1.2) is still subject of study. These commutation relations don't imply by themselves a minimal length, in the same way in which (1.1) doesn't prevent exact measures of position or momentum. Each edge of the quantum cell can be arbitrarily small if the other is accordingly large. Only their product is bounded from the below. For this reason in the case of (1.2) one should perhaps speak of a minimal area.

[3]In condensed matter certain systems are effectively 2-d in space.

In the limit of strong magnetic field $|\vec{B}| \gg m$ the equation (1.6) becomes

$$v^i = \frac{c}{e|\vec{B}|}\varepsilon^{ij} f^j(\vec{r}). \tag{1.7}$$

This last equation predicts that the particle moves always in a direction orthogonal to that of the external force (for example if the force is constant along the x axis, the particle moves with a constant velocity along the y axis. If the force vanishes, the particle stops).

Equation (1.6) is also simply obtained from the observation that the Lagrangian of the system is

$$\mathcal{L} = \frac{1}{2}mv^2 + \frac{e}{c}\vec{v} \cdot \vec{A} - V. \tag{1.8}$$

In the gauge $\vec{A} = (0, Bx)$, setting $m = 0$ we have

$$\mathcal{L} = \frac{e}{c}x\dot{y} - V(x, y) \tag{1.9}$$

This expression is of the form $\mathcal{L} = p\dot{q} - h(p, q)$ where $(\frac{e}{c}x, y)$ are a canonical pair. This implies that the system can be equivalently described by an Hamiltonian

$$H_0 = V \tag{1.10}$$

and variables satisfying (Poisson) relations

$$\left\{x^i, x^j\right\} = \frac{c}{e|\vec{B}|}\varepsilon^{ij}. \tag{1.11}$$

We see that, in the strong field limit, the original system is equivalent to another system in which no external field is present but with (Poisson) noncommuting coordinates. What happens in the strong field limit is that the Lagrangian becomes of the first order in time derivatives so that the original-commutative coordinate space can be viewed as an effective-noncommutative phase-space (1.11).

These arguments can be extended to the quantum level simply substituting the Poisson brackets with commutators. In this case (1.11) becomes

$$[x^i, x^j] = \frac{i\hbar c}{e|\vec{B}|}\varepsilon^{ij} \tag{1.12}$$

and the strong magnetic field limit corresponds to the projection on the lowest Landau level. This is a first example of canonical noncommutativity which involves constant (spacetime independent) commutation relations.

1.2.2 Canonical noncommutativity in string theories

Besides being relevant for electronic systems in external magnetic field, canonical noncommutativity has recently attracted a lot of interest since it emerges also from certain limits of

string theory. As an example we briefly discuss here bosonic open string in a flat space in presence of a constant B-field and Dp-branes [2]. If we indicate with r the rank of the matrix B_{ij}, we can assume that $r \leq p+1$ since the component of the field B not along the brane can be gauged away. Denoting with Σ the stringy worldsheet, the stringy action reads:

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma \left\{ g_{ij}\partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \varepsilon^{ab} \partial_a x^i \partial_b x^j \right\},$$

that can be also written as

$$S = \frac{1}{4\pi\alpha'} \int_\Sigma g_{ij}\partial_a x^i \partial^a x^j - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j, \tag{1.13}$$

where ∂_t is the tangent derivative along the string worldsheet boundary $\partial\Sigma$. The equations of motion for i along the Dp branes are:

$$g_{ij}\partial_n x^j + 2\pi i \alpha' B_{ij} \varepsilon^{ab} \partial_t x^j \Big|_{\partial\Sigma} = 0, \tag{1.14}$$

where ∂_n is a normal derivative to the worldsheet boundary $\partial\Sigma$.

In the strong field limit the propagator becomes

$$\left\langle x^i(\tau) x^j(\tau) \right\rangle = -\alpha^{'} G_{ij} \log(\tau - \tau^{'})^2 + \frac{i}{2}\theta^{ij}\epsilon(\tau - \tau^{'}) \tag{1.15}$$

This equation is incompatible with classical (commutative) coordinates and one can show [2] that this implies

$$[x^i(\tau), x^j(\tau)] = i\theta^{ij}. \tag{1.16}$$

In the strong field limit the first term in (1.13) becomes negligible so that the action takes the form

$$S = -\frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j, \tag{1.17}$$

which is purely topological. The boundary degrees of freedom become the dominant ones.

This is a second example of canonical noncommutativity. We observe that, though in a different context, the mechanism which generates this type of noncommutativity is always the same. A strong field renders the action topological (i.e. dependent only on a first derivative) and through the equation of motion the original commutative space transforms in an effective noncommutative phase-space.

1.2.3 Breakup of Lorentz symmetry in canonical spacetimes

We now want to comment briefly on the fate of the classical (Poincaré) symmetries in canonical noncommutative spacetime $[x_\mu, x_\nu] = i\theta_{\mu\nu}$. As already emphasized canonical noncommutativity is the weakest form of noncommutativity, in the sense that it involves coordinate independent commutation relations, and can be viewed as the zeroth-order approximation of the more general form $[x_\mu, x_\nu] = i\theta_{\mu\nu}(x)$. Here we suppose that the infinitesimal action of the

Poincaré group on the coordinates is not changed by noncommutativity (i.e. $M_j \triangleright x_\mu = [M_j, x_\mu]$, $N_j \triangleright x_\mu = [N_j, x_\mu]$, $P_\alpha \triangleright x_\mu = [P_\alpha, x_\mu]$). It can be easily verified that

$$M_j \triangleright [x_\mu, x_\nu] \neq M_j \triangleright (i\theta_{\mu\nu})\,, \tag{1.18}$$

$$N_j \triangleright [x_\mu, x_\nu] \neq N_j \triangleright (i\theta_{\mu\nu})\,, \tag{1.19}$$

$$P_j \triangleright [x_\mu, x_\nu] = P_j \triangleright (i\theta_{\mu\nu})\,. \tag{1.20}$$

These expressions show that canonical spacetime is not, in general, covariant under rotation and boosts while it is covariant under translations. Physically this means that there is not equivalence between different inertial observers (in the usual sense) since there is a preferred class of reference frames. Moreover in each frame of this class there is a preferred direction (which breaks rotations) but not a preferred origin (translations symmetry is preserved). It is also worth noticing that if noncommutativity does not involve time (i.e. $\theta_{\mu 0} = 0$) and one only considers infinitesimal boosts then

$$N_j \triangleright [x_\mu, x_\nu] = N_j \triangleright (i\theta_{\mu\nu})\,. \tag{1.21}$$

Actually in this case we have also covariance under finite non-relativistic boosts.

These mathematical results are in good agreement with the physical picture of canonical noncommutativity that comes from the description of charged particles in strong magnetic field. The covariance under boosts is lost since we are not considering transformation of the magnetic field ($\theta_{\mu\nu}$ in the noncommutative analogy), the rotational covariance is lost because the magnetic field selects a preferred direction while, the uniformity of the magnetic field (spacetime independent $\theta_{\mu\nu}$) ensures translational covariance.

1.3 Example of Lie-algebra noncommutative spacetime: κ-Minkowski spacetime

In this section we discuss a much-studied example of Lie-algebra noncommutative spacetime which turns out to play a role in attempts to describe Planck length L_p as a minimum wavelength λ. We first show how to implement such minimum-wavelength condition in a way that is compatible with the relativity postulates. Then we discuss the quantum group that can provide the corresponding mathematical framework and show that κ-Minkowski Lie-algebra spacetime is dual to this quantum group.

1.3.1 Planck length as a minimum wave-length

We have already discussed how there exist many different approaches to the unification of General Relativity with Quantum Mechanics which share the prediction of a minimum length. The existence of a minimum length is not by itself in contrast with the Poincaré symmetry, as the

quantization of the angular momentum is not in contrast with the invariance under continuous rotations. However if one wants to promote the Planck length to an invariant length, in the same sense in which the speed of light is an invariant velocity, one finds immediately inconsistence with Special Relativity. As the introduction of an invariant velocity scale necessarily induces some modifications in the Galilei group, so the introduction of a invariant length scale necessarily leads to modifications of the Lorentz group. Not all the elements of the Lorentz group will require modifications. For example, as intuition suggests, the idea of an invariant length scale is not in contrast with symmetry under spatial rotations therefore we should expect an unmodified rotation's sector. Deep modifications instead will be unavoidable in the boost sector, where the invariant length scale must appear as a deformation scale. In fact if the Planck scale is treated not as a (rescaled) coupling constant but as an observer-independent scale then one becomes in contrast with the Lorentz-Fitzgerald length contraction that prohibits observer-independent lengths.

Relativistic theories that admit two invariant scales, a velocity scale c and a length scale L_p were first introduced in the papers [6, 7] and have been extensively studied in the recent literature (see e.g. [8, 10, 27, 58, 59, 60]). These theories, commonly called "Doubly Special Relativity" (DSR) theories, can be formulated just in the same way in which Special Relativity is formulated: by introducing some corresponding postulates. One example of DSR postulates whose logical consistency has been analyzed in details in [61] is the following

- The laws of physics involve a fundamental velocity c and a fundamental length scale L_p.
- The value of the fundamental velocity scale c can be measured by each inertial observer as the speed of light with wavelength λ much larger than L_p (more rigorously, c is obtained as the $\lambda/L_p \to \infty$ limit of the speed of light)
- Each inertial observer can establish the value of L_p (same value for all inertial observers) by determining the dispersion relations for photons which takes the form $E^2 - c^2p^2 + f(E, p, L_p) = 0$, where the function f is the same for all the inertial observers and in particular all inertial observers agree on the leading L_p dependence of $f(E, p, L_p) \simeq cEp^2L_p$, i.e.
$$E^2 - c^2p^2 - cEp^2L_p = 0. \tag{1.22}$$

It is worth noticing that the notion of relativity in DSR theories is conceptually very similar to the notion of relativity in Galilean Relativity and in Special Relativity. The laws of physics are the same for all inertial observers. Inertial observers will not necessarily agree on the measured value of a given quantity but those relations among measurement results, which we call laws of physics, will hold for all inertial observers. For example, two inertial observers do not, in

general, agree on the value of the momentum of an electron, but if they measure energy and momentum of the electron, both observers will find that the measurements results satisfy the same dispersion relation. Among the laws of physics an important role is played by those which identify relativistic invariants quantities whose measurement gives the same result in all inertial frames.

In Galilean Relativity, for example, all inertial observers agree on the dispersion relations $E - p^2/(2m) = 0$. This relation does not involve any invariant scale other than the mass of the particle. In Special Relativity all inertial observers agree on the dispersion relation $E^2 - c^2p^2 - m^2c^4 = 0$, which involve only one invariant scale (c), other than the mass of the particle. In DSR theories all inertial observers agree on the relation of the type (1.22), which involves two invariant scale (c, L_p), other than the mass of the particle. All these theories are relativistic.

As already discussed the introduction of such postulates does not require to modify the rotation sector, but boosts do need to be modified. At the first order in L_p one can adopt the ansatz

$$N_i = i\left[cp_i + L_p\Delta_{1i}(\vec{p}^2, E)\right]\frac{\partial}{\partial E} + i\left[\frac{E}{c} + L_p\Delta_{2i}(\vec{p}^2, E)\right]\frac{\partial}{\partial p_i}, \tag{1.23}$$

and it is easy to verify that one has consistency with the dispersion relation (1.22) if Δ_{1i} and Δ_{2i} are such that the above expression takes the form

$$N_i = icp_i\frac{\partial}{\partial E} + i\left[\frac{E}{c} + L_p\left(\frac{E}{c}\right)^2 + \frac{L_p\vec{p}^2}{2}\right]\frac{\partial}{\partial p_i}. \tag{1.24}$$

Directly from (1.22) follows that $p^2 = E^2/(1+cEL_p)$. This deviation from the Special Relativistic dispersion relation $p = E$ implies that when $E \gtrsim 1/L_p$ the dependence of momentum on the energy change in the softest $p \simeq \sqrt{E/(cL_p)}$. This is a evidence of the momentum saturation that we will discuss more in detail when an all-orders form of f will have been obtained.

For a massless particle, if we retain the usual definition of speed[4], in the energy-momentum sector we get $v_\gamma = \frac{dE}{dp} = c(1 + \frac{L_p}{c}E)$ which predicts for sufficiently-energetic photons the possibility of $v_\gamma \gtrsim c$. Moreover the usual energy-momentum conservation rule needs to be modified in such way to be covariant under the new transformations rules. We will discuss more in details these point in the next section in the framework of κ-Poincaré quantum group.

1.3.2 κ-Poincaré Hopf algebras

It was observed in Refs. [62, 63] that the generators of (modified) boost and (unmodified) rotations constructed from DSR postulates correspond at the leading order in L_p to the Lorentz-sector generators of a well known quantum group: the bicrossproduct-basis κ-Poincaré Hopf algebra. The dispersion relation (1.22) if then the approximation at the leading order in L_p of

[4]We will discuss in the next chapter how this definition, beyond to be the more natural in the momentum sector, also is the one which comes from a proper analysis of the wave-packet motion in spacetime.

the Casimir of this Hopf algebra. It is useful for us to give here a brief review of the structure of κ-Poincaré Hopf algebra in the "bicrossproduct basis" [8, 7, 10] showing in particular the connection with κ-Minkowski noncommutative spacetime.

κ-Poincaré algebras were introduced in [6, 64] as one of the softest possible deformations of the usual Poincaré group as an Hopf algebra. It is defined in the algebraic sector by the commutation relations

$$[M_i, M_j] = i\epsilon_{ijk}M_k \qquad [M_i, P_j] = i\epsilon_{ijk}P_k, \tag{1.25}$$

$$[M_i, N_j] = i\epsilon_{ijk}N_k, \qquad [M_i, P_0] = 0, \tag{1.26}$$

$$[N_i, N_j] = -i\epsilon_{ijk}M_k, \qquad [P_\mu, P_\nu] = 0, \tag{1.27}$$

$$[N_j, P_0] = iP_j, \qquad [N_j, P_k] = i\delta_{jk}\left\{\frac{1-e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2}\vec{P}^2\right\} - i\lambda P_j P_k. \tag{1.28}$$

where $P_\mu = (P_0, P_i)$ are the four-momentum generators, M_k are the spatial rotation generators and N_i are the boost generators.

The algebraic relations (1.25-1.28) are accompanied by coalgebraic structures of the coproducts

$$\begin{aligned} \Delta(p_0) &= p_0 \otimes 1 + 1 \otimes p_0, \qquad & \Delta(p_j) &= p_j \otimes 1 + e^{-\lambda p_0} \otimes p_j, \\ \Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j, \qquad & \Delta(N_j) &= N_j \otimes 1 + e^{-\lambda p_0} \otimes N_j + \lambda\epsilon_{jkl}p_k \otimes M_l, \end{aligned} \tag{1.29}$$

and by the antipodes

$$\begin{aligned} S(N_j) &= -e^{\lambda p_0}N_j + \lambda e^{\lambda p_0}\epsilon_{jkl}p_k M_l, \qquad & S(M_j) &= -M_j, \\ S(p_j) &= -e^{\lambda p_0}p_j, \qquad & S(p_0) &= -p_0\ . \end{aligned} \tag{1.30}$$

A general description of Hopf algebras and in particular of their co-algebraic properties is given in Ref [65] (see also Appendix). Actually for the purposes of this thesis it suffices to notice that while the algebra describes products of generators the coalgebra essentially describes sums of generators. Nontrivial coproducts are related to non-Abelian addition law for the energy and momentum.

Now we analyze more closely the action of the Lorentz sector of this (quantum) Hopf algebra on the momentum sector. In a Lie-algebra context the action is obtained directly by the commutators (infinitesimal transformation) and by their exponentiation (finite transformation). In an Hopf algebra the action of a subalgebra on another subalgebra is generalized by the concept of action and coaction. From the definition of covariant (left)-adjoint action and from the

expressions of coproducts (1.29) and of antipodes (1.30) we have that

$$\begin{aligned}
M_i \triangleright M_j &= M_i^{(1)} M_j S(M_i^{(2)}) = [M_i, M_j] = i\epsilon_{ijk} M_k, \\
M_i \triangleright N_j &= M_i^{(1)} N_j S(M_i^{(2)}) = [M_i, N_j] = i\epsilon_{ijk} N_k, \\
M_i \triangleright P_\mu &= M_i^{(1)} P_\mu S(M_i^{(2)}) = [M_i, P_\mu] = i\epsilon_{i\mu k} P_k, \\
N_i \triangleright P_\mu &= N_i^{(1)} P_\mu S(N_j^{(2)}) = [N_i, P_\mu] = i\left[P_i \delta_{\mu 0} + \left(\frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2} P^2\right)\delta_{i\mu} - \lambda P_i P_k \delta_{k\mu}\right], \\
N_j \triangleright M_k &= N_j^{(1)} M_k S(N_j^{(2)}) = [N_j, M_k] + i\delta_{jk} \vec{P} \cdot \vec{M} - iP_j M_k, \\
N_j \triangleright N_k &= N_j^{(1)} N_k S(N_j^{(2)}) = [N_j, N_k] - i\lambda P_k N_j + \frac{i}{2}\epsilon_{ijk}\left(1 - e^{2\lambda P_0} + \lambda^2 P^2\right) M_l.
\end{aligned}$$

We observe that

- The action of rotations is undeformed.
- The action of boosts on the translation generators is deformed but it is still formulated through commutators.
- The action of boosts on boost/rotation generators is deformed and cannot be formulated through commutators.

From the above expression one can also easily write down the infinitesimal actions on a generic function, for example, of the momenta

$$\begin{aligned}
P_\mu \triangleright G(P) &= 0, \\
M_i \triangleright F(P) &= M_i^{(1)} F(P) S(M_i^{(2)}) = [M_i, F(P)] = -i\epsilon_{ijk} P_k \frac{\partial}{\partial P_l} F(P), \qquad (1.31) \\
N_i \triangleright F(P) &= N_i^{(1)} F(P) S(N_i^{(2)}) = [N_i, F(P)] = \\
&= i\left[P_i \frac{\partial}{\partial P_0} + \left(\frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2} P^2\right)\frac{\partial}{\partial P_i} - \lambda P_i P_k \frac{\partial}{\partial P_k}\right] F(P). \qquad (1.32)
\end{aligned}$$

These last expressions indicate that the action of the Lorentz sector $so_{1,3}$ on generic functions of the translation generators is still described by commutators. Also the usual Leibniz rule is satisfied

$$N_i \triangleright [F(P)G(P)] = [N_i \triangleright F(P)]G(P) + F(P)[N \triangleright G(P)]. \qquad (1.33)$$

Using the Leibniz rule (1.33) and (1.31-1.32) it is easy to calculate that for a generic finite transformation $e^{i(\vec{\theta}\cdot\vec{M}+\vec{\xi}\cdot\vec{N})}$ the action is

$$e^{i(\vec{\theta}\cdot\vec{M}+\vec{\xi}\cdot\vec{N})} \triangleright F(P) = e^{i(\vec{\theta}\cdot\vec{M}+\vec{\xi}\cdot\vec{N})} F(P) e^{-i(\vec{\theta}\cdot\vec{M}+\vec{\xi}\cdot\vec{N})}. \qquad (1.34)$$

The action (1.34) on functions of momenta is the usual one in spite of the non-trivial coalgebraic structure (1.29-1.30). This is due to the fact that the action of the Lorentz sector on

the momentum sector is still obtained through commutators. The "mass Casimir" of this Hopf algebra is

$$\mathcal{C}_\kappa(p) = \left(\frac{2}{\lambda}\sinh\frac{\lambda p_0}{2}\right)^2 - \vec{p}^2 e^{\lambda p_0}. \tag{1.35}$$

Since the action of Lorentz sector generator is still through commutators the Casimir $\mathcal{C}_\kappa$ preserves the property of being invariant under the covariant left adjoint action (1.31) and (1.32). Moreover the mass Casimir allows an unique, all-order determination of the function f introduced in (1.22), whereas DSR postulates fix it only to the lowest order in L_p. The same is true for the boost action (1.32) that is an all-order generalization of (1.24). Therefore we see that the DSR proposal is naturally realized in this Hopf-algebra quantum scheme and that all-order results in the deformation parameter can be obtained in this scheme. All-order expressions for the energy-momentum transformation rules between different inertial observers have been obtained in [63] and an all-order analysis of the scattering processes is reported in [66].

1.3.3 Phenomenology of κ-Poincaré

It is perhaps appropriate to pause for a few considerations regarding the phenomenological implications of κ-Poincaré kinematics. The mathematical formalism described so far already implies some profound modifications of the conventional special-relativistic kinematics framework. In particular from (1.35) one gets immediately [62] the important general conclusions that

- $E \to \infty$ when $|\vec{p}| \to 1/\lambda$,which means that there is a maximum momentum ($|\vec{p}| = 1/\lambda$).
- $v_\gamma(p) = \frac{dE}{dp} = (1 - p/\lambda)^{-1} = \exp(E/\lambda)$, which means that the speed of light tends to infinity when the momentum tends to the maximum allowed momentum.

It should be noticed that infinite velocities are allowed only when $E \to \infty$ so that an infinite amount of energy is needed to obtain a photon with infinite speed. Real photons have finite energy and therefore finite speed. Of course $v_\gamma \neq 1$ is a striking characteristic of this framework but for our "low-energy" particles ($E \ll 1/\lambda$) the effects are negligibly small [7, 10]. In practice this new framework is indistinguishable from the usual one at the presently-accessible energies. Therefore this striking prediction of the κ-Minkowski framework does not have troublesome phenomenological implications, but there is of course still an intense debate concerning the logical consistency of this scenario for $v_\gamma(p)$. In particular, it appears necessary to develop a new concept of causality. Actually, even before a full development of this new concept of causality, especially in cosmology there has been interest in the κ-Minkowski motivated idea of a light cone that effectively becomes wider as the energies available increase. Let us consider, for example, the horizon problem that is one of the main motivations for inflation theory. Horizon problem consists in the fact that zones of the sky which are angularly separated by more than a few degrees should be causally separated, whereas we observe significant large-scale homogeneity. The

analysis of Cosmic Microwave Background Radiation, for instance, indicates that the temperature is the same in all directions, with a precision of one part in 10^5. The hypothesis, predicted by DSR theories, of an energy-dependent speed of light might provide a simple explanation of this paradox [67], without recurring to the inflation theory. The greatly-energetic photons of the early stages of the universe in fact are predicted by DSR theories to have speeds high enough that they could causally connect zones of the universe that would otherwise, according to Special Relativity, be causally disconnected.

There are also other physical applications of κ-Poincaré kinematics that allow to explain certain paradoxes of physics based on the Special Relativity. A much studied example is the one in which one gives a κ-Poincaré/DSR description of the cosmic-ray paradox. According to the classical Poincaré symmetry of classical Minkowski spacetime ultra-high-energy-cosmic rays should loose energy interacting with the Cosmic Microwave Background Radiation by producing pions ($p + \gamma \to p + \pi$). Considering the typical energies ($E_{\gamma_{CMB}}$) of the CMRB photons and the typical distances from the Earth of the cosmic-ray sources, the assumption of validity of the Special Relativistic kinematic rules should lead to an upper (GZK) limit $E < 5 \cdot 10^{19} eV$ on the energy of the observed cosmic rays [68, 69]. Instead detection of several cosmic rays above the GZK limit has been reported [70]. One can calculate the thresholds for photopion production in the DSR scheme. Using the dispersion relation (1.22) and the energy-momentum conservation laws one finds

$$E > \frac{2m_p m_\pi + m_\pi^2}{4E_{\gamma_{CMB}}} + \lambda \frac{(2m_p + m_\pi)^3 m_\pi^3}{256 E_{\gamma_{CMB}}^4} \left(1 - \frac{m_p^2 + m_\pi^2}{(m_p + m_\pi)^2} \right), \tag{1.36}$$

where m_p (m_π) is the proton (pion) mass. Of course in the $\lambda \to 0$ limit one recovers the usual photopion-production threshold. We observe that in the correction term the smallness of λ (which, as mentioned, we expect to be of order $L_p \sim 10^{-33} cm$) is compensated by the huge ratios $m_p/E_{\gamma_{CMB}}, m_\pi/E_{\gamma_{CMB}}$. The result is that [21, 22] according to (1.36) even the observation of E$\sim 3 \cdot 10^{20} eV$ protons becomes possible providing an explanation to the observed cosmic rays.

Conceptually similar to the GZK paradox is the Markarian-501 paradox. High energy photons emitted by Markarian 501 with energies higher than 10TeV should collide with the Far Infrared Background Radiation (FIRBR) producing electron-positron pairs. Instead photons from Markarian 501 with energies higher than 20 TeV have been detected [71]. Markarian 501 paradox can be explained in a way analogous to the GZK paradox. The relevant process in this case is $\gamma + \gamma \to e^- + e^+$. The DSR-threshold for this process obtained from the dispersion relation (1.22) and energy-momentum conservation law is

$$E > \frac{m_e^2}{E_{\gamma_{FIRB}}} + \lambda \frac{m_\pi^6}{8E_{\gamma_{FIRB}}^4}. \tag{1.37}$$

Given the value involved in the energy $E_{\gamma_{FIRB}}$ of the FIRBR, the threshold is shifted up to $E \simeq 20TeV$ explaining the observations.

There are other physical situations, besides the ones mentioned so far in which κ-Poincaré kinematics might lead to detectable effects. Particularly promising in this sense appear to be the so-called time-of-flight studies in astrophysics. The velocity formula $v_\gamma(p)$ obtained in the κ-Poincaré framework predicts a difference in the times of arrival on Earth for two simultaneously-emitted photons

$$\Delta t_d \simeq \frac{\lambda \Delta E L}{c}, \tag{1.38}$$

where L is the source-Earth distance and ΔE is the difference between the energies of the two photons. Certain astrophysical objects produce very energetic photons that travel very large distances. From searches of a time-of-arrival difference of the type (1.38) it is possible to obtain bounds on the deformation parameter λ. Presently, the best bound is $\lambda \lesssim 500 L_p$ obtained by [18]. Future planned experiments, such as AMS [72] and GLAST [25, 73, 74], are expected to move this bound to the Planck scale (L_p) and beyond. In summary the phenomenology in the κ-Poincaré/DSR framework is rather rich: the new effects are small enough to agree with all available robust data, but large enough to provide candidate solutions for emerging experimental paradoxes and for testing in planned experiments.

1.3.4 κ-Minkowski Spacetime from κ-Poincaré duality

Having reassured the reader that the phenomenology of κ-Poincaré kinematics is acceptable and interesting, we go back to the analysis of the mathematical structure in order to identify a spacetime on which κ-Poincaré acts covariantly. Given the enveloping algebra of translation $T = (P_0, \vec{P})$, it is rather natural to take for the spacetime coordinate space its dual T^* that will also be an algebra on which T necessarily acts in a covariant way. The structure of the coordinate space T^* is univocally determined by the axioms of the Hopf algebra duality:

$$< t, xy > = < t_{(1)}, x >< t_{(2)}, y >, \tag{1.39}$$

$$< ts, x > = < t, x_{(1)} >< s, x_{(2)} > \qquad \forall t, s \in T, \ \forall x, y \in T^* \tag{1.40}$$

$$< p_\mu, x_\nu > = -i g_{\mu\nu}, \tag{1.41}$$

where $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $t_{(1)}$ and $t_{(2)}$ are introduced in the sense of the standard notation for the coproduct

$$\Delta t = \sum t_{(1)} \otimes t_{(2)}. \tag{1.42}$$

We see immediately from (1.40) that commutation rules in the momentum sector correspond to co-commutation rules in the coordinate sector

$$\Delta x_\mu = \mathbb{I} \otimes x_\mu + x_\mu \otimes \mathbb{I}. \tag{1.43}$$

The equation (1.39) and (1.41) can be used to define spacetime coordinates. One gets that

$$< p_i, x_0 x_j >= -\frac{i}{\kappa}\delta_{ij}, \tag{1.44}$$

$$< p_i, x_j x_0 >= 0, \tag{1.45}$$

from which it follows that $< p_i, [x_0, x_j] >= -\frac{1}{\kappa}\delta_{ij}$, and comparing with (1.41) one finds

$$[x_0, x_i] = -\frac{i}{\kappa}x_i, \tag{1.46}$$

$$[x_i, x_j] = 0. \tag{1.47}$$

This is the noncommutative spacetime called κ-Minkowski. We observe that spacetime noncommutativity in this case directly follows from the fact that the momentum sector, although being commutative, is not co-commutative, and from the duality relations pairing the two algebras (1.39-1.41).

1.3.5 Covariance of κ-Minkowski Spacetime

The duality between κ-Minkowski and (bicrossproduct) κ-Poincaré is related with the covariance, in the sense of Hopf algebras, of κ-Minkowski under κ-Poincaré action. The action of an element of the Lorentz sector w, on the coordinates is implicitly defined by the relation

$$\langle f(p), w\triangleright : g(x) :\rangle = \langle S(w) \triangleright f(p), : g(x) :\rangle\,, \tag{1.48}$$

where the action of $S(w)$ on functions of the momenta $f(p)$ is the (left) adjoint one (1.31-1.32) and :$g(x)$: is ordered in the form[5]

$$g(x) = \sum_{n_0 n_1 n_2 n_3} g_{n_1 n_2 n_3 n_0} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_0^{n_0}. \tag{1.49}$$

By using (1.31-1.32). and (1.48) one obtains the actions

$$M_j\triangleright : f(x) :=: -i\epsilon_{jkl}x_k\frac{\partial}{\partial x_l}f(x) :, \tag{1.50}$$

$$N_j\triangleright : f(x) :=: \left[ix_0\frac{\partial}{\partial x_j} + x_j\left(\frac{e^{2i\lambda\partial_t}-1}{2\lambda} + \frac{\lambda}{2}\bigtriangledown^2\right) - \lambda x_k\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}\right] f(x) : . \tag{1.51}$$

These actions are covariant in the Hopf algebra sense

$$M_j \triangleright [: f(x) :: g(x) :] = (M_j^{(1)}\triangleright : f(x) :)(M_j^{(2)}\triangleright : g(x) :), \tag{1.52}$$

$$N_j \triangleright [: f(x) :: g(x) :] = (N_j^{(1)}\triangleright : f(x) :)(N_j^{(2)}\triangleright : g(x) :). \tag{1.53}$$

We also observe that the action of boosts and rotations on the coordinates is the same as in Special Relativity

$$M_j \triangleright x_0 = 0, \qquad M_j \triangleright x_k = i\epsilon_{jkl}x_l, \qquad N_j \triangleright x_0 = ix_j, \qquad N_j \triangleright x_k = i\delta_{jk}x_0, \tag{1.54}$$

[5] We observe that every smooth function of noncommutative coordinates can be written in this form.

and can be described as an action through commutators.

Differences occur only in higher powers of the coordinates (1.50-1.51). Now we have to define the action of p_μ on the coordinate space. This action is defined by the (left) canonical action

$$p_\mu \overset{can}{\triangleright} : f(x) :=: f(x) :_{(1)} \langle p_\mu, : f(x) :_{(2)} \rangle =: -i\frac{\partial}{\partial x^\mu} f(x) : \tag{1.55}$$

so that a finite transformation reads

$$e^{iap} \overset{can}{\triangleright} : f(x) :=: f(x+a) :, \tag{1.56}$$

that is a simple translation[6]. This concludes our analysis of covariance of κ-Minkowski.

1.4 κ-Minkowski spacetime and canonical spacetime from general Lie-algebra spacetimes

In this section we observe that κ-Minkowski and canonical spacetime have the noticeable property that, among all the possible Lie-algebra spacetimes of the form

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} + iC^\alpha_{\mu\nu} x_\alpha, \tag{1.57}$$

they are selected from simple symmetry requirements. In particular if we demand invariance under undeformed rotations

$$M_j \triangleright [x_\mu, x_\nu] = M_j \triangleright \left(i\theta_{\mu\nu} + iC^\alpha_{\mu\nu} x_\alpha \right), \tag{1.58}$$

we obtain that it must be $\theta_{\mu\nu} = 0$ and

$$[x_i, x_j] = i\epsilon_{ijk} x_k, \qquad [x_i, x_0] = 0 \tag{1.59}$$

or

$$[x_i, x_0] = i\lambda x_i, \qquad [x_i, x_j] = 0. \tag{1.60}$$

The first spacetime (1.59) is known as fuzzy sphere, since $\mathcal{C} = x_1^2 + x_2^2 + x_3^2 + x_0^2$ is a Casimir of this algebra. In the second spacetime (1.60) we recognize κ-Minkowski spacetime.

On the other hand if we require invariance of (1.57) under undeformed translations

$$P_j \triangleright [x_\mu, x_\nu] = P_j \triangleright \left(i\theta_{\mu\nu} + iC^\alpha_{\mu\nu} x_\alpha \right) \tag{1.61}$$

we select

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \tag{1.62}$$

that is just canonical spacetime. This shows that among all the possible Lie-algebra spacetime the request of (undeformed) rotational invariance selects κ-Minkowski spacetime and the fuzzy

[6] It is worth noticing that defining the action of p_μ on $f(x)$ through the commutators of some quantum phase space (see for example (2.26)), this action can't be a translation.

sphere spacetime, whereas the request of (undeformed) translational invariance uniquely selects canonical spacetime.

To resume, with respect to the classical symmetries, we can construct the following table

Spacetime\Undef.Transf.	Rotations	Translations	Time-Translations	Boosts
κ-Minkowski	yes	no	yes	no
Canonical	no	yes	yes	no
Fuzzy sphere	yes	no	yes	no

Both κ-Minkowski and canonical spacetime are candidate noncommutative version of Minkowski spacetime[7] which in general preserve 4 undeformed classical symmetries (3 rotations + 1 time translation for κ-Minkowski, 3 spatial translations + 1 time translation for canonical spacetime). However, as discussed above, in addition to the 4 classical/undeformed symmetries, κ-Minkowski also enjoys 6 additional quantum-deformed symmetries, and its full symmetry structure is described by the 10 generators of κ-Poincaré Hopf algebra.

In the case of canonical noncommutativity the four classical symmetries are all there is (there are no additional deformed symmetries). This reflects the fact that canonical noncommutativity requires the support of a preferred class of inertial observers. In particular we know that canonical noncommutativity can be described as a commutative geometry in presence of a background (magnetic) field, and the presence of field allows the selection of a preferred class of inertial observer.

[7]Fuzzy sphere is not a good candidate since it represents a nonflat space.

Chapter 2

Waves in noncommutative Spacetimes

* In this chapter, after a brief review of the properties of waves in classical Minkowski spacetime, we analyze the concept of wave in a noncommutative spacetime. We show that the usual picture of propagating wave-packets can be implemented in a noncommutative context as well. We discuss how in the case of canonical noncommutativity this construction is very close to the commutative case: the phase velocity and the, physically more relevant, group velocity are the same as in the Minkowski spacetime in spite of the appearance of some extra (unobservable) phases that depend on the noncommutativity parameters. On the other hand, in κ-Minkowski spacetime, we find significant differences with respect to the commutative case. In particular we find that the description of the group velocity is still governed by the relation $v_g = dE(p)/dp$, but the dispersion relation $E(p)$ is significantly modified as discussed in the previous chapter.

2.1 Review of waves in Minkowski spacetime

Both in theories in Galilei spacetime and in theories in Minkowski spacetime the relation between the physical velocity of signals (the group velocity of a wave packet) and the dispersion relation is governed by the formula

$$v = \frac{dE}{dp} \ , \tag{2.1}$$

in components

$$v_j \equiv \frac{dx_j}{dt} = \frac{\partial E}{\partial p_j} = \frac{p_j}{p}\frac{\partial E}{\partial p} \ . \tag{2.2}$$

This is basically a result of the fact that our theories in Galilei and Minkowski spacetime admit Hamiltonian formulation. In classical mechanics this leads directly to

$$\frac{dx_j}{dt} = \frac{\partial H(p)}{\partial p_j} \ . \tag{2.3}$$

[0]* In this Chapter we discuss in detail the analysis reported more briefly in Ref. [46].

In ordinary quantum mechanics $\vec{x}$ and $\vec{p}$ are described in terms of operators that satisfy the commutation relations $[x_k, p_j] = i\delta_{jk}$, and in the Heisenberg picture the time evolution for the position operator is given by

$$\frac{dx_j(t)}{dt} = i[x_j(t), H]$$

Since $x_j \to \partial/\partial p_j$ and, again, $H \to E(p)$, also in ordinary quantum mechanics one finds $v = dE/dp$ (but in quantum mechanics v_j is the operator dx_j/dt and the group-velocity relation strictly holds only for expectation values).

Given a spacetime, the concept of group velocity can be most naturally investigated in the study of the propagation of waves. It is useful to review that discussion briefly. For simplicity we consider a classical 1+1-dimensional Minkowski spacetime. We denote by ω the frequency of the wave and by $k(\omega)$ the wave number of the wave. [Of course, $k(\omega)$ is governed by the dispersion relation, by the mass Casimir of the classical Poincaré algebra.] A plane wave is described by the exponential $e^{i\omega t - ikx}$. A wave packet is the Fourier transform of a function $a(\omega)$ which is nonvanishing in a limited region of the spectrum $(\omega_0 - \Delta,\, \omega_0 + \Delta\ )$:

$$\Psi_{(\omega_0,k_0)}(t,x) = \int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega) e^{i\omega t - ikx} d\omega\ .$$

The information/energy carried by the wave will travel at a sharply-specified velocity, the group velocity, only if $\Delta \ll \omega_0$. It is convenient to write the wave-packet as $\Psi_{(\omega_0,k_0)}(t,x) = A(t,x)e^{i\omega_0 t - ik_0 x}$, from which the definition of the wave amplitude $A(t,x)$ follows:

$$A(t,x) = \int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega) e^{i(\omega-\omega_0)t - i(k-k_0)x} d\omega \approx \int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega) e^{i(\omega-\omega_0)\left(t - \left[\frac{dk}{d\omega}\right]_0 x\right)} d\omega \tag{2.4}$$

The wave packet is therefore the product of the plane-wave factor $e^{i\omega_0 t - ik_0 x}$ and the wave amplitude $A(t,x)$. One can introduce a "phase velocity", $v_{ph} = \omega_0/k_0$, associated with the plane-wave factor $e^{i\omega_0 t - ik_0 x}$, but there is no information/energy that actually travels at this velocity (this "velocity" is a characteristic of a pure phase, with modulus 1 everywhere). It is the wave amplitude $A(t,x)$ that describes the time evolution of the energy/information actually carried by the wave packet. From (2.4) we see that the wave amplitude stiffly translates at velocity $v_g = d\omega_0/dk_0$, the group velocity. In terms of the group velocity and the phase velocity the wave packet can be written as

$$\Psi_{(\omega_0,k_0)}(t,x) = e^{ik_0(v_{ph}t - x)} \int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega) e^{i(\omega-\omega_0)(t - x/v_g)} d\omega\ .$$

In ordinary Minkowski spacetime the group velocity and the phase velocity both are 1 (in our units) for photons (light waves) travelling in vacuum. For massive particles or massless particles travelling in a medium $v_{ph} \neq v_g$. The causality structure of Minkowski spacetime guarantees that $v_g \leq 1$, whereas, since no information actually travels with the phase velocity, it provides no obstruction for $v_{ph} > 1$.

The main idea to extend the above construction to a noncommutative spacetime is that of writing the function of noncommutative variables as the inverse Fourier transform of a commutative energy-momentum-space function. Thus in general one writes

$$f(x) = \frac{1}{(2\pi)^4} \int dk^4 \tilde{f}(k) : \exp(ik^\mu x_\mu) :, \tag{2.5}$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$:

$$\tilde{f}(k) = \int d\alpha^4 f(\alpha) \exp(ik^\mu \alpha_\mu). \tag{2.6}$$

Here k and α are commuting variables while x are noncommuting variables. The function $: \exp(ik^\mu x_\mu) :$ must be consistent with the Fourier calculus and must reduce in the commutative limit to the usual $\exp(ik^\mu x_\mu)$. The advantage in using Fourier formulation is that one can do products of fields once it is known how to do products of the phases $: \exp(ik^\mu x_\mu) :$. This last ones are usually evaluated using the Baker-Campbell-Hausdorff formula.

2.2 Proposal of a description of wave-packets in Canonical Spacetime

We now try to extend the construction just outlined to waves in canonical spacetime. As a first step we have to define the phase $: \exp(ik^\mu x_\mu) :_\theta$. The simplest possible choice is

$$: \exp(ik^\mu x_\mu) :_\theta = \exp(ik^\mu x_\mu). \tag{2.7}$$

Once that single plane wave has been identified the next step is to consider products of waves. From the canonical noncommutativity relations we have that

$$\exp(ik^\mu x_\mu) \exp(ip^\mu x_\mu) = \exp(ik^\mu \theta_{\mu\nu} p^\nu) \exp[i(k^\mu + p^\mu) x_\mu]. \tag{2.8}$$

Waves packets are constructed summing plane waves in the usual way

$$\Psi_{(\omega_0, k_0)}(t, \vec{x}) = \int_{\omega_0 - \Delta}^{\omega_0 + \Delta} a(\omega) e^{i\omega t - i\, \vec{k} \cdot \vec{x}} d\omega \ .$$

Following the procedure outlined in the Minkowski case we find

$$\Psi_{(\omega_0, k_0)}(t, \vec{x}) \simeq \int_{\omega_0 - \Delta}^{\omega_0 + \Delta} a(\omega) e^{i\left[(\omega - \omega_0)t - \frac{d\vec{k}}{d\omega}(\omega - \omega_0)\vec{x}\right] + i\left[\omega_0 t - \vec{k}_0 \cdot \vec{x}\right]} d\omega \ ,$$

that is formally similar to (2.4). Using (2.8), defining $k^\mu \equiv (\omega - \omega_0, -\frac{d\vec{k}}{d\omega}(\omega - \omega_0))$ and $p^\mu \equiv (\omega_0, -\vec{k}_0)$ we get

$$\Psi_{(\omega_0, k_0)}(t, \vec{x}) \simeq \left[\int_{\omega_0 - \Delta}^{\omega_0 + \Delta} a(\omega) e^{i\left[(\omega - \omega_0)t - \frac{d\vec{k}}{d\omega}(\omega - \omega_0)\vec{x}\right]} \right] d\omega e^{-ik^\mu \theta_{\mu\nu} p^\nu} e^{i\left[\omega_0 t - \vec{k}_0 \cdot \vec{x}\right]}.$$

As in the Minkowski case we can individuate an amplitude here

$$A(t,\vec{x}) = \left[\int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega) e^{i\left[(\omega-\omega_0)t - \frac{d\vec{k}}{d\omega}(\omega-\omega_0)\vec{x}\right]}\right], \tag{2.9}$$

and write the wave packet as

$$\Psi_{(\omega_0,k_0)}(t,\vec{x}) \simeq e^{ik^\mu\theta_{\mu\nu}p^\nu} e^{i\vec{k}_0\cdot\left[\vec{v}_{ph}t-\vec{x}\right]} \left[\int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega) e^{i\frac{d\vec{k}}{d\omega}(\omega-\omega_0)[\vec{v}_g t-\vec{x}]} d\omega\right], \tag{2.10}$$

where we have defined $v_g = \frac{d\omega}{dk}$, and $v_{ph} = \frac{\omega_0}{k_0}$. At this point some remarks are in order. A first observation regards the overall phase factor $e^{ik^\mu\theta_{\mu\nu}p^\nu}$ which appears in (2.10). This factor doesn't give any modification of the group velocity and is just an overall phase velocity. A second observation regards the fact that we would have reached the same conclusion by factorizing the phase contribution to the right:

$$\Psi_{(\omega_0,k_0)}(t,x) \simeq \left[\int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega) e^{i\frac{dk}{d\omega}(\omega-\omega_0)[v_g t-x]} d\omega\right] e^{-ik^\mu\theta_{\mu\nu}p^\nu} e^{ik_0\left[v_{ph}t-x\right]}. \tag{2.11}$$

The only difference being now that the overall phase factor acquires an opposite sign with respect to the previous case. However the results for v_g, v_{ph} do not depend on this sign.

These observations seem to suggest that in the case of canonical noncommutativity there is no observable modification of the in vacuum propagation of wave packets.

2.3 The challenge of Waves in κ-Minkowski Spacetime

In this section we try to construct wave packets in κ-Minkowski spacetime following the line of the previous sections. We use the Fourier calculus in noncommutative spaces developed in [75] for a proper definition of the wave packet. This assure the right covariance properties of the packets under κ-Poincaré transformations and allow us to get phase velocity and group velocity.

2.3.1 Differential calculus and Fourier calculus in κ-Minkowski

The only consistently-developed differential calculus [65] on the enveloping algebra of κ-Minkowski is

$$\partial_j :f(x):=: \frac{\partial f(x)}{\partial x_j}: \tag{2.12}$$

$$\partial_0 :f(x):=: \frac{e^{i\lambda\frac{\partial}{\partial t}}-1}{i\lambda} f(x) :=: \frac{f(\vec{x},t+i\lambda)-f(\vec{x},t)}{i\lambda}: \tag{2.13}$$

The notation $:f(x):$, conventional in the κ-Minkowski literature, is reserved for time-to-the-right-ordered[1] functions of the noncommutative coordinates. The standard symbolism adopted

[1]In κ-Minkowski spacetime (with its commuting space coordinates and nontrivial commutation relations only when the time coordinate is involved), it is easy to see that the natural functional calculus should be introduced in terms of time-to-the-right-ordered functions or (the equivalent alternative of) intuitive rules for time-to-the-left-ordered functions. In other noncommutative spacetimes the choice of ordering may not be so obvious.

in Eqs. (2.12)-(2.13) describes noncommutative differentials in terms of familiar actions on commutative functions. The symbols "∂_j" and "∂_0" refer to elements of the differential calculus on κ-Minkowski, while the symbols "$\partial/\partial x_j$" and "$\partial/\partial t$" act as ordinary derivatives on a time-to-the-right-ordered function of the κ-Minkowski coordinates. For example, Eq. (2.12) states that in κ-Minkowski $\partial_x(xt) = t$ and $\partial_x[xt^2+2i\lambda xt-\lambda^2 x+x^2 t] = t^2+2i\lambda t-\lambda^2+2xt$, *i.e.* ∂_x acts as a familiar x-derivative on time-to-the-right-ordered functions. Of course, the κ-Minkowski commutation relations impose that, if derivatives are standard on time-to-the-right-ordered functions, derivatives must be accordingly modified for functions which are not time-to-the-right ordered. For example, since $\partial_x(xt) = t$ and $\partial_x(x) = 1$ (the functions xt and x are time-to-the-right ordered), also taking into account the κ-Minkowski commutation relation $xt = tx - i\lambda x$, one can obtain the x-derivative of the function tx, which must be given by $\partial_x(tx) = t + i\lambda$. Similarly, one finds that $\partial_x[t^2 x + x^2 t] = t^2 + 2i\lambda t - \lambda^2 + 2xt$ (in fact, using the κ-Minkowski commutation relations one finds that $t^2 x + x^2 t = xt^2 + 2i\lambda xt - \lambda^2 x + x^2 t]$).

The time derivative described by Eq. (2.13) has analogous structure, with the only difference that the special role of the time coordinate in the structure of κ-Minkowski spacetime forces [65] one to introduce an element of discretization in the time direction: the time derivative of time-to-the-right-ordered functions is indeed standard (just like the x-derivative of time-to-the-right-ordered functions is standard), but it is a standard λ-discretized derivative (whereas the x-derivative of time-to-the-right-ordered functions is a standard continuous derivative).

A central role in the κ-Minkowski functional calculus is played by the ordered exponentials:

$$e^{-i\vec{q}\vec{x}}e^{iq_0 t} \ , \tag{2.14}$$

where $\{q_j, q_0\}$ are four real numbers and $\{x_j, t\}$ are κ-Minkowski coordinates. These ordered exponentials enjoy a simple property with respect to the generators p_μ of translations of the κ-Minkowski coordinates:

$$\left\langle p_\mu, e^{-i\vec{q}\vec{x}}e^{iq_0 t}\right\rangle = q_\mu \ . \tag{2.15}$$

We also note that, using the κ-Minkowski commutation relations, one finds the relation

$$e^{-i\vec{q}\vec{x}}e^{iq_0 t} = \exp\left(iq_0 t - i\vec{q}\vec{x}\frac{\lambda q_0}{1-e^{-\lambda q_0}}\right) \tag{2.16}$$

which turns out to be useful in certain applications.

The ordered exponentials $e^{-i\vec{q}\vec{x}}e^{i\omega t}$ also play the role of plane waves in κ-Minkowski since on the mass-shell (*i.e.* $\mathcal{C}_\kappa(q_0,\vec{q}) = M^2$) they are solutions [10] of the relevant wave (deformed Klein-Gordon) equation:

$$(\Box - M^2)\left[e^{-i\vec{q}\vec{x}}e^{iq_0 t}\right] = 0 \tag{2.17}$$

where $\Box = \partial_\mu\partial^\mu L^{-1}$ is the κ-deformed D'Alembert operator, properly defined [65, 10] in terms of the so-called "κ-Minkowski shift operator" L

$$L : f(\vec{x},t) := e^{-\lambda p_0} \triangleright : f(\vec{x},t) : = : f(\vec{x}, t+i\lambda) :$$

The ordered exponentials are also the basic ingredient of the Fourier theory on κ-Minkowski. This Fourier theory [65] is constructed in terms of the canonical element $\sum_i e_i \otimes f^i$, where $\{e_i\}$ and $\{f^j\}$ are dual bases, which satisfy the relation $< e_i, f^j > = \delta_i^j$. On the basis of (1.39-1.41) one finds that the canonical element is

$$\psi_{(q_0,\vec{q})}(t,\vec{x}) = \sum_{n_0,n_1,n_2,n_3}^{0,\infty} \frac{(-iq_1x_1)^{n_1}}{n_1!} \frac{(-iq_2x_2)^{n_2}}{n_2!} \frac{(-iq_3x_3)^{n_3}}{n_3!} \frac{(iq_0t)^{n_0}}{n_0!} = e^{-i\vec{k}\vec{x}} e^{i\omega t} \tag{2.18}$$

The canonical element (2.18) retains the notable feature that, if we define the transform $\tilde{f}(q)$ of an ordered function $:f(x):$ through

$$:f(x): = \int \tilde{f}(q)\, e^{-i\vec{q}\vec{x}} e^{iq_0t}\, \frac{e^{3\lambda q_0} d^4q}{(2\pi)^4} \ ,$$

the choice of the integration measure $e^{3\lambda q_0}$ and the definition (1.48) of the actions of boosts/rotations on the coordinates guarantee that

$$w \triangleright :f(x): = \int \Big(S(w) \triangleright \tilde{f}(q) \Big)\, e^{-i\vec{q}\vec{x}} e^{iq_0t}\, \frac{e^{3\lambda q_0} d^4q}{(2\pi)^4}$$

for each $w \in U(so_{1,3})$. This is a relevant property because it implies that under a finite transformation both $:f:$ and $\tilde{f}$ change, but they remain connected by the Fourier-transform relations. The action of a transformation on the x is equivalent to the inverse transformation on the q. This is exactly what happens in the classical-Minkowski case ($\lambda = 0$), through the simple relation

$$f(x) \mapsto f_\Lambda(x) = \int \tilde{f}(\Lambda^{-1}q)\, e^{iqx}\, \frac{d^4q}{(2\pi)^4}.$$

In κ-Minkowski the action of boosts does not allow description in terms of a matrix Λ_μ^ν, but it is still true that the action of a transformation on the x is equivalent to the "inverse transformation" on the q (where, of course, here the "inverse transformation" is described through the antipode).

2.3.2 Group velocity in κ-Minkowski

The elements of κ-Minkowski functional analysis we reviewed in Section 2 allow us to implement a consistent deformation of the analysis that applies in commutative Minkowski spacetime, here reviewed in the preceding subsection. In order to present specific formulas we adopt the κ-Minkowski functional analysis based on time-to-the-right-ordered noncommutative functions, but the careful reader can easily verify that the same result for the group velocity is obtained adopting the time-to-the-left ordering prescription.

We are little concerned with the concept of phase velocity (which is not a physical velocity). In this respect we just observe that the phase velocity should be a property of the κ-Minkowski plane wave

$$\psi_{(\omega,\vec{k})} = e^{-i\vec{k}\vec{x}} e^{i\omega t} \ , \tag{2.19}$$

and, since the κ-Minkowski calculus is structured in such a way that the properties of time-to-right-ordered functions are just the ones of the corresponding commutative function, this suggests that the relation

$$v_{ph} = \frac{\omega}{k} \tag{2.20}$$

should be valid.

But let us focus on the more significant (physically meaningful) analysis of group velocity. Our starting point is the wave packet

$$\Psi_{(\omega_0,\vec{k}_0)} = \int e^{-i\vec{k}\cdot\vec{x}} e^{i\omega t} d\mu.$$

In this equation (2.3.2) for simplicity we denote with $d\mu$ an integration measure which includes the spectrum of the packet. In fact, the precise structure of the wave packet is irrelevant for the analysis of the group velocity: it suffices to adopt a packet which is centered at some $(\omega_0, \vec{k}_0)$ (with $(\omega_0$ and $\vec{k}_0)$ related through Eq. (1.35), the dispersion relation, the mass Casimir, of the classical Poincaré algebra) and has support only on a relatively small neighborhood of $(\omega_0, \vec{k}_0)$, *i.e.* $\omega_0 - \Delta\omega \leq \omega \leq \omega_0 + \Delta\omega$ and $\vec{k}_0 - \Delta\vec{k} \leq \vec{k} \leq \vec{k}_0 + \Delta\vec{k}$.

Next, in order to proceed just following the same steps of the familiar commutative-spacetime case, we should factor out of the integral a "pure phase" with frequency and wavelength fixed by the wave-packet center: $(\omega_0, \vec{k}_0)$. Consistently with the nature of the time-to-the-right-ordered functional calculus the phase e^{ik_0x} will be factored out to the left and the phase $e^{-i\omega_0 t}$ will be factored out to the right:

$$\Psi_{(\omega_0,\vec{k}_0)} = e^{-i\vec{k}_0\cdot\vec{x}} \left[\int e^{-i\Delta\vec{k}\cdot\vec{x}} e^{i\Delta\omega t} d\mu\right] e^{i\omega_0 t} \tag{2.21}$$

This way to extract the phase factor preserves the time-to-the-right-ordered structure of the wave $\Psi_{(\omega_0,\vec{k}_0)}$, and therefore, also taking into account the role that time-to-the-right-ordered functions have in the κ-Minkowski calculus, should allow an intuitive analysis of its properties.

From (2.21) one recognizes the κ-Minkowski group velocity as

$$v_g = \lim_{\Delta\omega\to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} \ , \tag{2.22}$$

just as in Galilei and Minkowski spacetime. Just as one does in commutative Minkowski spacetime, the integral can be seen as the amplitude of the wave, the group velocity v_g is the velocity of translation of this wave amplitude, which be meaningfully introduced only in the limit of narrow packet (small $\Delta\omega$ and $\Delta\vec{k}$).

Notice that

$$e^{-i\Delta\vec{k}\vec{x}} e^{i\Delta\omega t} = \exp\left(i\Delta\omega t - i\Delta\vec{k}\vec{x}\frac{\lambda\Delta\omega}{1 - e^{-\lambda\Delta\omega}}\right) \ , \tag{2.23}$$

and

$$\left[\exp\left(i\Delta\omega t - i\Delta\vec{k}\vec{x}\frac{\lambda\Delta\omega}{1 - e^{-\lambda\Delta\omega}}\right)\right]_{\Delta\omega\to 0} = \exp\left(i\Delta\omega t - i\Delta\vec{k}\vec{x}\right) \ , \tag{2.24}$$

and therefore the evaluation of the velocity of translation of this wave amplitude turns out to be independent of the way in which the exponentials are arranged (but this is an accident due to the fact that for small $\Delta\omega$ and $\Delta\vec{k}$ one finds that $[e^{-i\Delta\vec{k}\vec{x}}, e^{i\Delta\omega t}] = 0$.

2.4 Comparison with previous analyses

Because of the mentioned interest in the phenomenological implications [17, 18, 12, 13, 14, 74, 25], the introduction of group velocity in κ-Minkowski has been discussed in several studies. In the large majority of these studies the concept of group velocity was not introduced constructively (it was not a result obtained in a full theoretical scheme: it was just introduced through an *ad hoc* relation). This appeared to be harmless since the *ad-hoc* assumption relied on the validity of the relation $v_g = dE/dp$, which holds in Galilei spacetime and Minkowski spacetime (and for which the structure of κ-Minkowski appears to pose no obstacle).

Taking as starting point the approach to κ-Minkowski proposed in Ref. [10], we have here shown through a dedicated analysis that the validity of $v_g = dE/dp$ indeed follows automatically from the structure of κ-Minkowski and of the associated functional calculus.

At this point it is necessary for us to clarify which erroneous assumptions led to the claims reported in Refs. [27, 28, 76], which questioned the validity of $v_g = dE(p)/dp$ in κ-Minkowski.

2.4.1 Tamaki-Harada-Miyamoto-Torii analysis

It is rather easy to compare our analysis with the study reported by Tamaki, Harada, Miyamoto and Torii in Refs. [28, 76]. In fact, Ref. [28] explicitly adopted the same approach to κ-Minkowski calculus that we adopted here, with Fourier transform and functional calculus that make direct reference to time-to-the-right-ordered functions. Also the scheme of analysis is analogous to ours, in that it attempts to derive the group velocity from the analysis of the time evolution of a superposition of plane waves. However, the κ-Minkowski functional calculus was applied inconsistently in Ref. [28]: at the stage of the analysis were one should factor out the phases $e^{-i\vec{k}_0\cdot\vec{x}}$ and $e^{i\omega_0 t}$ from the wave amplitude (as we did in Eq. (2.21)) Ref. [28] does not proceed consistently with the time-to-the-right-ordered functional calculus. Of course, as done here, in order to maintain the time-to-the-right-ordered form of the wave packet it is necessary to factor out the phases $e^{-i\vec{k}_0\cdot\vec{x}}$ and $e^{i\omega_0 t}$ respectively to the left and to the right, as we did here. Instead in Ref. [28] both phases are factored out to the left leading to a form of the wave packet which is not time-to-the-right ordered. In turn this leads to the erroneous conclusion that $v_g(k) \neq d\omega(k)/dk$, *i.e.* $v_g(p) \neq dE(p)/dp$.

This inconsistency with the ordering conventions is the key factor that affected Ref. [28] failure to reproduce $v_g(p) \neq dE(p)/dp$, but for completeness we note here also that Ref. [28] leads readers to the erroneous impression that in order to introduce the group velocity in κ-

Minkowski one should adopt the approximation

$$e^{-i\vec{k}\vec{x}}e^{i\omega t} \sim e^{-i\vec{k}\vec{x}+i\omega t} , \tag{2.25}$$

for generic values of ω and $\vec{k}$. Actually, unless ω and $\vec{k}$ are very small, this approximation is very poor: it only holds in zeroth order in the noncommutativity scale λ and therefore it does not describe reliably the structure of κ-Minkowski (since it fails already in leading order in λ, it does not even reliably characterize the main differences between classical Minkowski and κ-Minkowski). As we showed here there is no need for the approximation (2.25) in the analysis of the group velocity of a wave packet in κ-Minkowski.

2.4.2 κ-Deformed phase space

As discussed in the preceding Subsection, it is very easy to compare our study with the study reported in Ref. [28], since both studies adopted the same approach. We must now provide some guidance for the comparison with the study reported by Kowalski-Glikman in Ref. [27]. Also this comparison is significant for us since Ref. [27], like Ref. [27], questioned the validity of the relation $v_g = dE(p)/dp$, which instead emerged from our analysis.

Our approach to κ-Minkowski, which originates from techniques developed in Refs. [8, 10], is profoundly different from the one adopted in Ref. [27]. In fact, the differences start off already at the level of the action of κ-Poincaré generators on κ-Minkowski coordinates. The actions we adopted are described in Section 2. They take a simple form on time-to-the-right ordered functions, but they do not allow description as a "commutator action" on generic ordering of functions in κ-Minkowski. Instead in Ref. [27] the action of the κ-Poincaré generators on κ-Minkowski coordinates was introduced in fully general terms as a commutator action. This would allow to introduce a "phase-space extension" of κ-Minkowski [27]

$$[x_0, x_j] = i\lambda x_j, \qquad [p_0, x_0] = -i, \qquad [p_k, x_j] = i\delta_{jk}e^{-\lambda p_0}, \qquad [p_j, x_0] = [p_0, x_j] = 0. \tag{2.26}$$

Taking this phase space (2.26) as starting point, Kowalski-Glikman then found, after a rather lengthy analysis, that "massless particles move in spacetime with universal speed of light" c [27], in conflict with the relation $v_g = dE(p)/dp$ and the structure of the mass Casimir (1.35). Kowalski-Glikman argued that this puzzling conflict with the structure of the mass Casimir might be due to a mismatch between the mass-Casimir relation, $E(p, m)$, and the dispersion relation, $\omega(k, m)$: the puzzle could be explained [27] if the usual identifications $k \sim p$ and $\omega \sim E$ were to be replaced by $k \sim pe^{\lambda E}$ and $\omega \sim sinh(\lambda E)/\lambda + \lambda p^2 e^{\lambda E}/2$.

We observe that the correct explanation of the puzzling result obtained by Kowalski-Glikman is actually much simpler: the commutator action (2.26) adopted in Ref. [27], in spite of the choice of symbols p_j,x_k, <u>cannot</u> describe the action of "momenta" p_j on coordinates x_k. Momenta should generate translations of the coordinates, which requires that they may be represented

as derivatives of functions of the coordinates, but the commutator action $[p_k, x_j] = i\delta_{jk}e^{-\lambda p_0}$ clearly does not allow to represent p_k as a derivative with respect to the x_k coordinate, because of the spurious factor $e^{-\lambda p_0}$. Similarly, those commutation relations do not allow to represent the x_k coordinate as a derivative with respect to p_k, and therefore in a Hamiltonian theory, with Hamiltonian H, one would find

$$\dot{x}_j \sim [x_j, H] \neq \frac{dH}{dp_j} , \tag{2.27}$$

and this is basically the reason for the puzzling result $v_g(p) \neq dE(p)/dp$ obtained in Ref. [27]. Kowalski-Glikman finds a function $v_g(p)$ but this function cannot be seen as describing the relation between velocity and momentum, since the "p" symbol introduced in (2.26) does not generate translations of coordinates, and therefore "p" is not a momentum.

Chapter 3

Quantum Field Theories on Canonical Spacetimes

* In this chapter we discuss the construction of quantum field theory in noncommutative spacetimes. We observe that the analysis of quantization in κ-Minkowski spacetime is still at a preliminary stage. Thus we focus mainly on the quantization in canonical spacetime which has been extensively analyzed in literature. The most common strategy to construct a quantum field theory on noncommutative space is through the Weyl map which allows to introduce certain structures in noncommutative spacetimes in analogy with the corresponding structures in commutative spacetime. The product of functions in the commutative space is described through a deformed "*" or "Moyal" product of corresponding functions. Here we will discuss the main features of the quantum field theories obtained through this procedure. In particular we focus on the so-called IR-UV mixing which is one of the most interesting and distinguishing features of these theories.

3.1 Weyl Quantization in canonical and κ-Minkowski spacetime

3.1.1 Weyl Quantization in the phase space of ordinary quantum mechanics

Weyl quantization is a technique used (see e.g.[77, 78]) to describe quantum mechanics using the phase-space of classical mechanics. The general idea consists in defining a mapping between the algebra of operators on the quantum phase space and the algebra of functions on the classical phase space. This map must of course be compatible with the product in the noncommutative algebra of quantum operators. To achieve this goal one has to deform the usual product of commutative functions on the phase space in a new product called Weyl-Moyal product. In quantum mechanics the Weyl map is given by:

$$W : F(x,p) \to W[F] = \frac{1}{(2\pi)^4} \int d\alpha^4 d\beta^4 \tilde{F}(\alpha,\beta) \exp i(\alpha\hat{x} + \beta\hat{p}) \tag{3.1}$$

[0]* In this Chapter we review the issue of QFT construction in noncommutative spacetimes.

where $\tilde{F}(\alpha,\beta)$ is the Fourier transform of $F(x,p)$. If we consider products of functions of the quantum phase space we obtain

$$W[F]W[G] = \frac{1}{(2\pi)^8}\int d\alpha^4 d\beta^4 d\alpha'^4 d\beta'^4 \tilde{F}(\alpha,\beta)\tilde{F}(\alpha',\beta')\exp i(\alpha\hat{x}+\beta\hat{p})\exp i(\alpha'\hat{x}+\beta'\hat{p}). \quad (3.2)$$

Since it holds that $W[F]W[G] \neq W[FG]$, it follows that, as introduced, the Weyl map does not preserve the usual product of commutative functions on the phase space. However the same formula (3.2) suggests how to modify the product to have compatibility. We notice that (3.2) can be easily evaluated using the commutation relation $[\hat{x}_i,\hat{p}_j] = i\hbar\delta_{ij}$ and the Baker-Campbell-Hausdorff expansion

$$\exp i(\alpha\hat{x}+\beta\hat{p})\exp i(\alpha'\hat{x}+\beta'\hat{p}) = \exp i[(\alpha+\alpha')\hat{x}+(\beta+\beta')\hat{p}]\exp[\frac{i}{2}\hbar(\alpha\beta'-\alpha'\beta)], \quad (3.3)$$

from which follows that

$$W[F]W[G] = \frac{1}{(2\pi)^8}\int d\alpha^4 d\beta^4 d\alpha'^4 d\beta'^4 \tilde{F}(\alpha,\beta)\tilde{F}(\alpha'-\alpha,\beta'-\beta)e^{i[\alpha\hat{x}+\beta\hat{p}]}e^{\frac{i}{2}\hbar[\alpha\beta'-\alpha'\beta]} = \quad (3.4)$$

$$= \frac{1}{(2\pi)^4}\int d\alpha^4 d\beta^4 \exp i[\alpha\hat{x}+\beta\hat{p}]\tilde{A}(\alpha,\beta) \quad (3.5)$$

where

$$\tilde{A}(\alpha,\beta) = \frac{1}{(2\pi)^4}\int d\alpha'^4 d\beta'^4 \tilde{F}(\alpha,\beta)\tilde{F}(\alpha'-\alpha,\beta'-\beta)\exp\frac{i}{2}\hbar[\alpha\beta'-\alpha'\beta]. \quad (3.6)$$

Thus one is led to introduce a new product "$*$" between commutative functions on the classical phase space such that

$$F(x,p)*G(x,p) = W^{-1}\left(W[F]W[G]\right) = \frac{1}{(2\pi)^4}\int d\alpha^4 d\beta^4 \exp i[\alpha x+\beta p]\tilde{A}(\alpha,\beta). \quad (3.7)$$

One can easily verify that this $*$ product has the following differential expression on the phase space

$$F(x,p)*G(x,p) = F(x,p)\exp\frac{i}{2}\hbar\left(\frac{\overleftarrow{\partial}}{\partial x}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overrightarrow{\partial}}{\partial x}\frac{\overleftarrow{\partial}}{\partial p}\right)G(x,p). \quad (3.8)$$

The Weyl-Moyal $*$ product so defined is associative but noncommutative and in particular one has that commutation relation is properly mapped

$$[x,p]_* = x*p - p*x = i\hbar. \quad (3.9)$$

3.1.2 Weyl quantization for canonical noncommutativity

To deal with the quantization of spacetime one can repeat for spacetime the procedure just outlined for the phase-space of quantum mechanics. A first point to notice is the relevance of the choice of the Weyl map and in particular the choice of the order of noncommuting coordinates in the expression that defines the Weyl map. Different choices lead to different Weyl-Moyal products. A second important point is that this proposal of quantization, eventually,

takes into account only the quantization of spacetime. Quantization of the phase-space must anyway be implemented separately. We will assume the conservative hypothesis that it can be implemented with the usual strategies as, for instance, that of the path-integral formulation. We start considering Weyl quantization for canonical spacetime. In this case one can define the map between functions on noncommutative spacetime and functions on commutative spacetime as

$$W_\theta : F(x) \to W_\theta[F] = \frac{1}{(2\pi)^4} \int d\alpha^4 \tilde{F}(\alpha) \exp(i\alpha_\mu \hat{x}^\mu), \tag{3.10}$$

where $\tilde{F}(\alpha)$ is the Fourier transform of $F(x)$. Now let us consider products of functions of canonical spacetime

$$W_\theta[F] W_\theta[G] = \frac{1}{(2\pi)^8} \int d\alpha^4 d\beta^4 \tilde{F}(\alpha) \tilde{G}(\beta) \exp i(\alpha_\mu \hat{x}^\mu) \exp i(\beta_\mu \hat{x}^\mu). \tag{3.11}$$

Al already discussed it holds that

$$\exp(i\alpha^\mu x_\mu) \exp(i\beta^\mu x_\mu) = \exp(i\alpha^\mu \theta_{\mu\nu} \beta^\nu) \exp[i(\alpha^\mu + \beta^\mu) x_\mu], \tag{3.12}$$

thus

$$W_\theta[F] W_\theta[G] = \frac{1}{(2\pi)^8} \int d\alpha^4 d\beta^4 \tilde{F}(\alpha - \beta) \tilde{G}(\beta) \exp(i\alpha^\mu \theta_{\mu\nu} \beta^\nu) \exp[i\alpha^\mu x_\mu] = \tag{3.13}$$

$$= \frac{1}{(2\pi)^4} \int d\alpha^4 \exp[i\alpha^\mu x_\mu] \tilde{A}(\alpha), \tag{3.14}$$

where

$$\tilde{A}(\alpha) = \frac{1}{(2\pi)^4} \int d\beta^4 \tilde{F}(\alpha - \beta) \tilde{G}(\beta) \exp(i\alpha^\mu \theta_{\mu\nu} \beta^\nu). \tag{3.15}$$

Again it is natural to introduce a Moyal $\star$-product

$$F(x) \star G(x) = \frac{1}{(2\pi)^4} \int d\alpha^4 \exp[i\alpha^\mu x_\mu] \tilde{A}(\alpha) = F(x) \exp[\frac{i}{2} \theta_{\mu\nu} \frac{\overleftarrow{\partial}}{\partial x^\mu} \frac{\overrightarrow{\partial}}{\partial x^\nu}] G(x). \tag{3.16}$$

In this way we automatically have that $W_\theta[F] W_\theta[G] = W_\theta[F \star G]$. Also this Moyal $\star$-product is associative but it is noncommutative. One can easily verify that the commutation relations are properly mapped:

$$[x_\mu, x_\nu]_\star = x_\mu x_\nu - x_\nu x_\mu = i\theta_{\mu\nu}. \tag{3.17}$$

For the product of several waves one finds

$$e^{ik_1 x} \star e^{ik_2 x} \star \ldots \star e^{ik_n x} = e^{ix \sum_{i=1}^n k_j - \frac{i}{2} \sum_{i,j=1, i<j}^n k_i \theta k_j}, \tag{3.18}$$

whereas complex coniugation gives

$$\overline{F \star G} = \overline{F} \star \overline{G}. \tag{3.19}$$

One can also introduce integrals directly in the noncommutative space by defining

$$\int d\hat{x}^4 : \exp(ik_\mu \hat{x}^\mu) :_\theta \equiv \delta^4(k), \tag{3.20}$$

where $\hat{x}^\mu$ are noncommutative coordinates. Using the above formula one can calculate integrals of every noncommutative function as follows

$$\int d\hat{x}^4 W[F] = \frac{1}{(2\pi)^4}\int d\alpha^4 \tilde{F}(\alpha)\int d\hat{x}^4 \exp(i\alpha_\mu \hat{x}^\mu) = \tilde{F}(0) = \int dx^4 F(x). \tag{3.21}$$

It also results that

$$\int d\hat{x}^4 W[F]W[G] = \int dx^4 F(x) \star G(x). \tag{3.22}$$

Therefore canonical noncommutative integrals defined by (3.20) reduce to the $\star$-deformed commutative integrals. Using these relations and the Fourier transform properties it is easy to verify the following properties of the star product under integration:

1. $\int dx^4 F(x) \star G(x) = \int dx^4 F(x)G(x)$,

2. $\int dx^4 F_1(x) \star F_2(x) \star ... \star F_n(x) = \int dx^4 F_n(x) \star F_1(x) \star ... \star F_{n-1}(x)$,

3. $\int dx^4 F_1(x) \star F_2(x) \star ... \star F_n(x) = \int \frac{dp_1^4 ... dp_n^4}{(2\pi)^{n-1}} \widetilde{F}_1(p_1)...\widetilde{F}_n(p_n)\delta^4(p_1 + ... + p_n)e^{-\frac{i}{2}\sum_{i,j,i<j}^{1...n} k_i\theta k_j}$.

The first property implies that under integration the $\star$-product of two functions is the same as the common product of functions. The second properties says that under integrations the $\star$-product is invariant under cyclic permutations. The third property relates (under integrations) $\star$-products of functions with products of their Fourier transforms.

One can also introduce a differential calculus on canonical noncommutative spacetime defining derivatives through the formulas

$$[\hat{\partial}_i, x_j] = \delta_{ij}, \tag{3.23}$$

$$[\hat{\partial}_i, \hat{\partial}_j] = 0. \tag{3.24}$$

These formulas are compatible with the spacetime commutation relation

$$\hat{\partial}_k[\hat{x}_i, \hat{x}_j] = \hat{\partial}_k \theta_{ij} = 0. \tag{3.25}$$

It is also easy to verify that

$$\hat{\partial}_k W_\theta[F] = W_\theta[\partial_k F], \tag{3.26}$$

and that

$$\int d\hat{x}^4 \hat{\partial}_k W_\theta[F] = \int d\hat{x}^4 W_\theta[\partial_k F] = \int dx^4 \partial_k F = 0. \tag{3.27}$$

With these definitions of integrals and derivatives, it is possible to define noncommutative versions of the usual commutative actions. An example that we will analyze in detail in the following sections is provided by the noncommutative scalar theory whose action is

$$S = \int d\hat{x}^4 \left[\frac{1}{2}\left(\hat{\partial}_\mu W_\theta[\varphi]\right)^2 + \frac{m^2}{2}W_\theta[\varphi]^2 + \frac{\lambda}{4!}W_\theta[\varphi]^4\right]. \tag{3.28}$$

In terms of commutative functions this action is written as:

$$S = \int dx^4 \left[\frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi \star \varphi \star \varphi \star \varphi\right]. \tag{3.29}$$

3.1.3 Weyl quantization for κ-Minkowski spacetime

In the case of κ-Minkowski spacetime the procedure is exactly the same. The first step is to fix the Weyl map

$$W_\kappa : F(x) \to W_\kappa[F] = \frac{1}{(2\pi)^4} \int d\alpha^4 \tilde{F}(\alpha) : \exp i(\alpha_\mu x^\mu) :_\kappa, \tag{3.30}$$

which means to choose the ordering in $: \exp i(\alpha x) :_\kappa$. On the basis of the considerations already discussed in the previous chapter one is naturally led to adopt the ordering prescription

$$: \exp i(\alpha_\mu x^\mu) :_\kappa = \exp i(\alpha_i x^i) \exp i(\alpha_0 x^0). \tag{3.31}$$

Now let us consider the products of functions in the quantum spacetime

$$W_\kappa[F] W_\kappa[G] = \frac{1}{(2\pi)^8} \int d\alpha^4 d\beta \tilde{F}(\alpha) \tilde{G}(\beta) : \exp i(\alpha_\mu x^\mu) :_\kappa : \exp i(\beta_\mu x^\mu) :_\kappa . \tag{3.32}$$

Again we have that $W[F]W[G] \neq W[FG]$, which means that the Weyl map does not preserve the usual product of commutative function. Using the commutation relation of κ-Minkowski and the Baker-Campbell-Hausdorff expansion one finds

$$: \exp i(\alpha_\mu x^\mu) :_\kappa : \exp i(\beta_\mu x^\mu) :_\kappa = : \exp i[(\alpha_\mu \dot{+} \beta_\mu) x^\mu] :_\kappa, \tag{3.33}$$

where $\dot{+}$ is such that

$$\alpha_\mu \dot{+} \beta_\mu \equiv \delta_{\mu 0}(\alpha_0 + \beta_0) + (1 - \delta_{\mu 0})[\alpha_\mu + \exp(\beta_0/\kappa)\beta_\mu]. \tag{3.34}$$

Using the above expression we can rewrite (3.32) as

$$W_\kappa[F] W_\kappa[G] = \frac{1}{(2\pi)^8} \int d\alpha^4 d\beta^4 \tilde{F}(\alpha) \tilde{G}(\beta) : \exp i[(\alpha_\mu \dot{+} \beta_\mu) x^\mu] :_\kappa . \tag{3.35}$$

Therefore the Weyl-Moyal product for κ-Minkowski is

$$F(x) \overset{\kappa}{*} G(x) = W_\kappa^{-1}(W_\kappa[F] W_\kappa[G]) = F(x) \exp[i x^\mu \sigma_\mu \left(\overleftarrow{\partial}_x, \overrightarrow{\partial}_x\right)] G(x), \tag{3.36}$$

where

$$\sigma_0(\alpha, \beta) = \alpha_0 + \beta_0 \tag{3.37}$$

$$\sigma_i(\alpha, \beta) = (1 - \delta_{\mu 0})[\alpha_\mu + \exp(\beta_0/\kappa)\beta_\mu] \tag{3.38}$$

The $\overset{\kappa}{*}$-product is associative but it is noncommutative. And in particular we have that the commutation relations are properly mapped

$$[x_0, x_i]_{\overset{\kappa}{*}} = x_0 \overset{\kappa}{*} x_i - x_i \overset{\kappa}{*} x_0 = \lambda x_i, \tag{3.39}$$

$$[x_i, x_j]_{\overset{\kappa}{*}} = x_i \overset{\kappa}{*} x_j - x_j \overset{\kappa}{*} x_i = 0, \tag{3.40}$$

and waves combine in agreement with (3.33)

$$\exp i(\alpha_\mu x^\mu) \overset{\kappa}{*} \exp i(\beta_\mu x^\mu) = \exp i(\alpha_\mu \dot{+} \beta_\mu) x^\mu. \tag{3.41}$$

Again one can introduce integrals directly in the noncommutative space by defining

$$\int d\hat{x}^4 : \exp(ik_\mu \hat{x}^\mu) :_\kappa \equiv \delta^4_\kappa(k), \tag{3.42}$$

where $\hat{x}^\mu$ are the noncommutative coordinates. From the above formula one can calculate the integrals of every noncommutative function in κ-Minkowski spacetime as follows

$$\int d\hat{x}^4 W[F] = \frac{1}{(2\pi)^4} \int d\alpha^4 \tilde{F}(\alpha) \int d\hat{x}^4 \exp(i\alpha_\mu \hat{x}^\mu) = \tilde{F}(0) = \int dx^4 F(x). \tag{3.43}$$

It results that

$$\begin{aligned} \int d\hat{x}^4 W[F]W[G] &= \int dx^4 F(x) \overset{\kappa}{*} G(x) = \int d\alpha^4 d\beta^4 F(\alpha) G(\beta) \int dx^4 \exp i(\alpha_\mu \dot{+} \beta_\mu) x^\mu = \\ &= \int d\alpha^4 d\beta^4 F(\alpha) G(\beta) \delta^4_\kappa(\alpha_\mu \dot{+} \beta_\mu). \end{aligned} \tag{3.44}$$

One can also verify the useful formula

$$\int d\alpha^4 F(\alpha) \delta^4_\kappa(\alpha_\mu \dot{+} \beta_\mu) = \mu(p_0) F(\dot{-}\beta), \tag{3.45}$$

where $\mu(p_0)$, plays the role of a integration measure. We observe that most of the properties under integration of the canonical $\star$-product are not shared by the κ-Minkowski $\overset{\kappa}{*}$-product. Most notably the integral $\overset{\kappa}{*}$ product of functions is not symmetric under cyclic permutations of the argument functions and in the integral of $\overset{\kappa}{*}$-product of two functions the dependence on the noncommutative parameter does not disappear.

We also observe that Fourier momenta combine nonlinearly in the arguments of the δ- function. This will produce profound implications in the construction of the quantum field theory.

3.1.4 Functional formalism in noncommutative space

The basic hypothesis of the most popular approach to QFT in noncommutative spacetime is that the Hilbert space is not modified by the noncommutativity so that the physicals relevant information is still encoded in the Green functions defined as

$$G^{(n)}(x_{1,\dots,}x_n) = \langle 0 | T\{\phi(x_1)...\phi(x_n)\} | 0 \rangle = \int D\phi\ \phi(x_1)...\phi(x_n) \exp iS_\theta(\phi). \tag{3.46}$$

All the dependence on the noncommutativity parameters is contained in the new action $S_\theta(\phi)$. We observe that this popular approach may be also viewed as an assumption of "minimality": the action is modified but the entire procedure that from the action lead us to the physical predictions is assumed to be unaffected by noncommutativity. It is not inconceivable

that a meaningful QFT in NCST might also be introduced through a more profound modification of the conceptual structure adopted in commutative-spacetime frameworks, but such alternative formulations remain so far largely unexplored.

One can obtain the Green functions (3.46) from a generating functional defined as usual

$$Z[J] = \int D\phi \exp i \left\{ S(\phi) + \int dx^4 J(x)\phi(x) \right\}, \tag{3.47}$$

where we notice that the source $J(x)$ is coupled to the field $\phi(x)$ in the usual way. In this scheme all the machinery of the ordinary field theory can be carried on. For example we define as usual the generating functional for the connected Green's function $W[J]$

$$Z[J] = e^{iW[J]}, \tag{3.48}$$

and the 1PI-effective action

$$\Gamma[\varphi] = W[J] - \int J(x)\varphi(x)dx^4, \tag{3.49}$$

where $\varphi(x) = \langle 0 |\phi(x)| 0 \rangle = \frac{\delta W[J]}{\delta J(x)}$.

Also the usual perturbative expansions (weak coupling expansion, loop expansion ecc.) in this framework hold unchanged.

3.2 Scalar $\lambda\varphi^4$-theory in canonical spacetime

3.2.1 Action, functional derivatives and equation of motion

According to the arguments discussed in the previous chapter noncommutative scalar theory is simply obtained from the usual commutative scalar theory with the only prescription of substituting every product of the fields with a $\star$-product. As already emphasized the substitutions do not change the quadratic part of the action. For example in the case of scalar $\lambda\varphi^4$-theory the action can be written as

$$S = \int dx^4 \left\{ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi \right\}. \tag{3.50}$$

As usual one obtains the equation of motion from the request of stationarity of the action

$$\frac{\delta S}{\delta\phi} = 0, \tag{3.51}$$

where we have adopted the usual definition of functional derivative. We stress that we are now working in a commutative space of functions, in which however just the product is deformed.

From the action (3.50) and the fact that

$$\frac{\delta\phi\star\phi\star\phi\star\phi(x)}{\delta\phi(y)} = \delta^4(x-y)\star\phi\star\phi\star\phi + \phi\star\delta^4(x-y)\star\phi\star\phi + \tag{3.52}$$

$$+ \phi\star\phi\star\delta^4(x-y)\star\phi + \phi\star\phi\star\phi\star\delta^4(x-y) \tag{3.53}$$

one easily obtains the equation of motion

$$(\Box + m^2)\phi = \frac{\lambda}{6}\phi \star \phi \star \phi. \tag{3.54}$$

It is worth noticing some differences for the solutions of this equation of motion with respect to the commutative case. For example in noncommutative $\lambda\varphi^4$ one finds solitonic solutions whereas in classical (4d-commutative) $\lambda\varphi^4$-scalar theory Derrick theorem prohibits the existence of all finite energy classical solutions. Derrick theorem is based on the observation that if all lengths are scaled as $L \to \alpha L$ both the kinetic and the potential energies decrease so that no finite-size minimum can exist. It is not surprising that this argument fails in presence of a characteristic length scale, such as $\sqrt{\theta}$ in the canonical noncommutative case. In fact it was shown in Ref. [79] that for sufficiently large θ stable solitons can exist in the noncommutative theory. Mathematically this is due to the fact that equations of the type $\lambda\phi\star\phi+\phi = 0$, which is a typical example of solitonic equation for the scalar theory, admit non-trivial solutions (whereas the corresponding equation $\lambda\phi^2 + \phi = 0$ for the commutative case only has constant solutions).

3.2.2 Feynman diagrams

The analysis of Feynman diagrams in theories constructed on canonical noncommutative spacetime is particularly interesting since important differences emerge with respect to the commutative counterpart. We start considering a generic scalar interaction given by the vertex

$$S_{int} = \frac{\lambda}{4!}\int dx^4 \underbrace{\phi \star ... \star \phi}_{n} = \frac{\lambda}{4!}\int \frac{dp_1^4 ... dp_n^4}{(2\pi)^{n-1}}\tilde{\phi}(p_1)...\tilde{\phi}(p_n)\delta^4(p_1 + ... + p_n)\exp(-\frac{i}{2}\sum_{i,j=1,i<j}^{n} p_i\theta p_j). \tag{3.55}$$

A first observation is that the usual δ-function of energy-momentum conservation is still present so that the usual energy-momentum conservation rules, at each vertex, hold unchanged. This is in agreement with what suggested by the analysis of symmetries of canonical spacetime in Section 1.4. The important differences with respect to the usual Feynman rules, of the corresponding commutative interaction, is the appearance of the phase factor $V(p_1, ..., p_n) = \exp(-\frac{i}{2}\sum_{i,j=1,i<j}^{n} p_i\theta p_j)$. This factor must be taken into account and in particular one must preserve the order of the lines attached to each vertex. The order of the lines attached at each vertex is not in general important, but in noncommutative case it is crucial.

Planar diagrams and nonplanar diagrams

Noncommutative theories have the feature that the total contribution to Green functions, while still symmetries under momenta exchange, is obtained summing vertices which are not themselves symmetric under the exchange of momenta entering the vertex. This means that in the usual perturbative expansion one must pay particular care, even in a single-field theory,

keeping track of the order of the momenta attached at each vertex[1]. Particularly important with respect to this issue is the distinction between diagrams that are or nonplanar in the noncommutative-theory sense. This noncommutative theory concept of planarity is best introduced through an example.

Let us consider the diagrams contributing to the self-energy of $\lambda\varphi^4$ scalar theory. We distinguish the line incoming into a vertex by the numbers 1,2,3,4. Given the first external line (of associate momentum p) we can attach the vertex by one of the lines 1,2,3 or 4. In a vertex without any symmetry under exchange of the momenta, these different choices correspond to different contributions. In our case, thanks to the symmetry under cyclical exchanges, all these choices gives the same contributions. We have only to multiply the results times the number of possible choices (four in this case). Now let us consider the case in which the external momentum p is attached to the vertex line-1 (Fig.3.1).

Figure 3.1: External momentum and ordered vertex.

There are of course three possible ways to attach the second external line of momentum k to the vertex. If we choose the vertex line-2 or the vertex line-4 (Fig.3.2) we can connect by a propagator the remaining vertex lines in a way to obtain planar diagrams

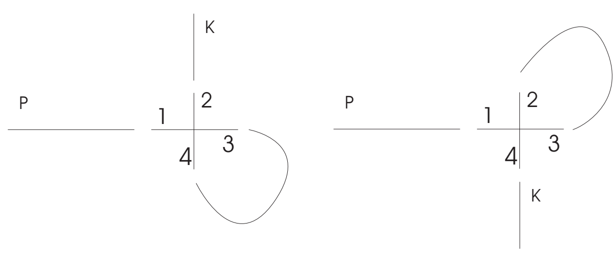

Figure 3.2: Planar connections of the vertex lines.

These diagrams all have will give the same $\star$-product induced phase contribution $(\exp(-\frac{i}{2}p\theta p) = 1)$. Instead, if we attach the second external line to the vertex line-3, (Fig.3.3) the only way we have to connect the remaining vertex lines 2,4 is through a propagator which intersects at least one line

This means that this diagrams is nonplanar. Its phase factor is $\exp(-\frac{i}{2}p\theta k)$. We will discuss the implications of these phases in the next sections. Finally for the self energy we have 4

[1] This operation is of course without consequences in the commutative counterpart thanks to the symmetry of the vertex.

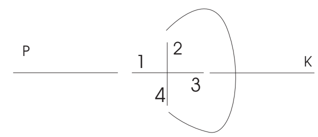

Figure 3.3: Nonplanar connection of the vertex lines.

nonplanar diagrams with the phase factor $\exp(-\frac{i}{2}p\theta k)$ and 8 planar diagrams with phase factor 1.

More in general nonplanar diagrams are those diagrams than cannot be, in any way, drawn in a plane without intersecting (at least) two lines. Planar diagrams are those diagrams that are not nonplanar. It the usual Feynman diagrammatic planar diagrams, though always well distinct from the nonplanar ones, give the same numerical contributions. This is essentially due to the invariance of the vertices under generic permutation of the momenta. In theories in which the order of the lines incoming in each vertex is important, in general, different combinations give different contributions. However, in the case of the theory we are considering we have the important properties of the momenta-conservation at each vertex and the invariance of the vertex under cyclic permutation of the attached momenta. With this properties one can prove that the phase factor is the same for all the possible complex-planar diagrams and reads

$$V(p_1, ..., p_n) = \exp(-\frac{i}{2} \sum_{i<j<n} p_i \theta p_j). \tag{3.56}$$

It depends only on the order of the external momenta: it is insensitive to the internal structure of the graph. This implies that the contribution of a planar graph is precisely the same of the corresponding diagrams of the corresponding commutative theory multiplied by $V(p_1, ..., p_n)$. The eventual divergencies also are the same and they can be treated similarly to the commutative case.

We said that nonplanar diagrams cannot be drawn in a plane in such way that propagators do not cross each other. It is rather easy to see that any nonplanar graph, for each crossing of the momenta k_i and k_j, will acquire an extra phase

$$\exp ik_i \times k_j, \tag{3.57}$$

in addition to the phase associated with the ordering of external momenta. Therefore one has for the complete phase factor of a nonplanar graph

$$V(p_1, ..., p_n) \exp(-\frac{i}{2} \sum_{i<j<n} C_{ij} k_i \theta k_j), \tag{3.58}$$

where $V(p_1, ..., p_n)$ is as in (3.56) and C_{ij} is an intersection matrix that counts the number of times that the i-th line (internal or external) crosses the over the j-th line. Crossing are counted

as positive if p_i crosses p_j with p_j moving to the left. There is not an one to one correspondence between graphs and C_{ij} matrices since different way of drawing the graph will lead to different intersections. However all these matrices give the same Feynman integral. We see that in the case on nonplanar graph the θ-dependence cannot be factorized out as in the planar case. This θ-dependence is the cause of profound differences in the behavior of the diagrams. In particular one has that the phase factor improve the UV convergence of the diagrams and one might expect that, with the exception of divergent planar subgraphs, all nonplanar graphs to be finite. We will see however that new IR divergences appear. Moreover as a consequence of the internal phase factor nonplanar diagrams vanishes in the $\theta \to \infty$ limit (strong commutativity limit). In this limit the theory is the sum of planar diagrams only (planar limit).

3.2.3 One-loop 1PI effective action and the IR/UV Mixing.

As an explicit example of calculation with this new diagrammatic we consider the 1PI two-point function. At the lowest order we have that it is simply the inverse propagator

$$\Gamma^2_0(p) = p^2 + m^2, \tag{3.59}$$

that is unchanged. At one loop one has to sum the diagrams of Figs.3.4-3.5

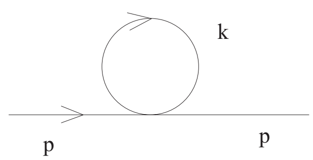

Figure 3.4: Planar tadpole contribution.

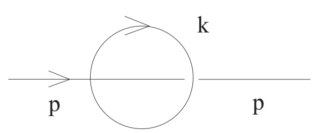

Figure 3.5: Nonplanar tadpole contribution.

The first (Fig.3.4) is a planar diagram while the second (Fig.3.5) is a nonplanar diagram. We observe that in the $\theta \to 0$ limit the two diagrams become the same (planar) diagrams with the right commutative combinatorial factor. Their contributions are respectively

$$\Gamma^2_{\rm pl}(p) = \frac{\lambda}{3}\int \frac{dk^4}{(2\pi)^2}\frac{1}{k^2+m^2}, \tag{3.60}$$

$$\Gamma^2_{\rm npl}(p) = \frac{\lambda}{6}\int \frac{dk^4}{(2\pi)^2}\frac{\exp[ip\theta k]}{k^2+m^2}. \tag{3.61}$$

We observe that in the $\theta \to 0$ limit the integrands of the two diagrams become equal. The planar contribution (3.60) is the same as in the commutative case (up to a numerical factor)

and it is quadratically divergent in the ultraviolet sector, whereas the nonplanar contribution (3.61) is finite thanks to the oscillation produced by the phase in the integrand. To evaluate explicitly (3.60-3.61) it is useful to use the Schwinger parametrization

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2+m^2)}. \tag{3.62}$$

Substituting (3.62) in (3.60) and (3.61) and integrating in the Gaussian variables k one obtains

$$\Gamma^2_{\mathrm{pl}}(p) = \frac{\lambda}{48\pi^2}\int_0^\infty d\alpha \frac{e^{-\alpha m^2 - \frac{1}{\alpha\Lambda^2}}}{\alpha^2}, \tag{3.63}$$

$$\Gamma^2_{\mathrm{npl}}(p) = \frac{\lambda}{96\pi^2}\int_0^\infty d\alpha \frac{e^{-\alpha m^2 - \frac{p\theta^2 p}{\alpha} - \frac{1}{\alpha\Lambda^2}}}{\alpha^2}. \tag{3.64}$$

We have introduced in both diagrams explicitly the cut-off Λ. We observe that the planar contribution behaves as usual (i.e. it has a leading-quadratic divergence for large momenta), whereas the nonplanar contribution is finite even after removing the cutoff. The evaluation of the above expressions gives

$$\Gamma^2_{\mathrm{pl}}(p) = \frac{\lambda}{48\pi^2}(\Lambda^2 - m^2\ln(\frac{\Lambda^2}{m^2}) + O(1)), \tag{3.65}$$

$$\Gamma^2_{\mathrm{npl}}(p) = \frac{\lambda}{96\pi^2} m^2 \sqrt{\frac{\Lambda^2_{\mathrm{eff}}}{m^2}} K_1\left(\frac{m}{\Lambda^2_{\mathrm{eff}}}\right) = \frac{\lambda}{96\pi^2}(\Lambda^2_{\mathrm{eff}} - m^2\ln(\frac{\Lambda^2_{\mathrm{eff}}}{m^2}) + O(1)), \tag{3.66}$$

where

$$\Lambda^2_{\mathrm{eff}} = \frac{1}{1/\Lambda^2 + p\theta^2 p} \tag{3.67}$$

and $K_1(x)$ is a modified-Bessel function of the first kind.

We observe that the contributions coming from the nonplanar diagrams remain finite in the $\Lambda \to \infty$ limit and that in the same limit $\Lambda^2_{\mathrm{eff}} \to (p\theta^2 p)^{-1}$. Noncommutativity regularizes the divergences in the nonplanar diagrams but the planar ones remain divergent as in the commutative case. Moreover unexpected[2] (IR-)divergences appear in the limit of vanishing momenta $p \to 0$. Explicitly up to one-loop the two-point effective action reads

$$\Gamma^2(\phi) = \int dp^4 \phi(p)\phi(-p)\frac{1}{2}\left\{p^2 + m_R^2 + \right. \tag{3.68}$$

$$\left. -\frac{\lambda}{96\pi^2}\left(\frac{1}{1/\Lambda^2 + p\theta^2 p} - m_R^2 \ln(\frac{1}{m_R^2(1/\Lambda^2 + p\theta^2 p)}) + O(1)\right) + O(\lambda^2)\right\}, \tag{3.69}$$

where

$$m_R^2 = m^2 + \frac{\lambda^2}{48\pi^2}(\Lambda^2 - m^2\ln(\frac{\Lambda^2}{m^2}) + O(1)) \tag{3.70}$$

is the renormalized mass. It is worth noticing that

[2]In the usual quantum picture of spacetime a caracteristic length scale α introduce new features only at lenght scale smaller than α. Instead in the case of canonical noncommutativity, where the new length scale is $\sqrt{\theta}$, corrections are introduced at lenght scale larger than $1/\sqrt{\theta}$.

- In the limit $p\theta^2 p \gg 1/\Lambda^2$ one recovers $\Lambda^2_{\text{eff}} \simeq \frac{1}{p\theta^2 p}$ and the effective action (3.68) becomes

$$\Gamma^2(\phi) = \int dp^4 \phi(p)\phi(-p)\frac{1}{2}\left\{p^2 + m_R^2 - \frac{\lambda^2}{96\pi^2}\left(\frac{1}{p\theta^2 p} - m_R^2 \ln(\frac{1}{m_R^2 p\theta^2 p}) + O(1)\right) + O(\lambda^2)\right\} \tag{3.71}$$

- Instead if $p\theta^2 p \ll 1/\Lambda^2$ one obtains $\Lambda_{\text{eff}} \simeq \Lambda$ and the commutative expression

$$\Gamma^2(\phi) = \int dp^4 \phi(p)\phi(-p)\frac{1}{2}\left\{p^2 + \widetilde{m}_R^2 + O(\lambda^4)\right\} \tag{3.72}$$

  is recovered, where $\widetilde{m}_R^2 = m^2 + \frac{\lambda^2}{96\pi^2}(\Lambda^2 - m^2 \ln(\frac{\Lambda^2}{m^2}) + O(1))$ is the commutative renormalized mass.

We see that in (3.71) there are singularities in the infrared ($p \to 0$) limit, involving quadratic and logarithmic poles. Surprisingly the same poles persist even in the limit $\theta \to 0$. This means that after the renormalization of the theory (i.e. after the removal of the cutoff Λ) the limit $\theta \to 0$ does not give back the commutative theory. If we instead work at fixed cutoff Λ the limit $\theta \to 0$ can always be taken and it always gives back the commutative theory. The fact that the ultra-violet (UV) limit $\Lambda \to \infty$ and the infra-red (IR) limit $p \to 0$ do not commute is a manifestation of a mixing of the ultraviolet degrees of freedom with the infrared ones. In literature this mixing is known as IR/UV mixing [34].

On the validity of perturbative expansion

We want to report some observations regarding the validity of the perturbative expansion in these theories [34]. The point is that even if the 1-loop contributions are all of order λ, with respect to the tree level, they diverge in the $p \to 0$ limit. This might motivate some skepticism toward the validity of the perturbative expansion. The one-loop contribution becomes greater than the tree level one when

$$p^2 + m_R^2 \lesssim \frac{\lambda}{p\theta^2 p}. \tag{3.73}$$

At the n-th order the divergent dependence in p of the nonplanar diagram may be read from the dependence on Λ of the planar diagrams. Thus one expects at the n-th order the leading singularities in p are of the type

$$\Gamma_n^2(p) \approx \frac{\lambda^n}{p\theta^2 p}\left[\ln(m_R^2 p\theta^2 p)\right]^{n-1}. \tag{3.74}$$

This higher order contributions are as large as the first order when

$$\frac{\lambda^n}{p\theta^2 p}\left[\ln(m_R^2 p\theta^2 p)\right]^{n-1} \approx \frac{\lambda}{p\theta^2 p}, \tag{3.75}$$

which yields

$$m_R^2 p\theta^2 p < e^{-\frac{c}{\lambda^2}}, \tag{3.76}$$

where c is a dimensionless constant. Therefore the one loop approximation is valid for

$$O(e^{-\frac{c}{\lambda}}) < m_R^2 p\theta^2 p < O(\lambda), \tag{3.77}$$

which means that the range of momenta in which the loop expansion is meaningful is exponentially small in terms of the inverse of the coupling constant.

One-loop vertex function

The tree-level vertex is easily obtained from (3.55) by functional derivatives. It reads

$$\Gamma^4(p,q,r,s) = \frac{\lambda}{4!}\delta^4(p+q+r+s)V_s(p,q,r,s), \tag{3.78}$$

where

$$V_s(p,q,r,s) = \frac{1}{3}\left[\cos\left(\frac{p\theta^2 s - q\theta^2 r}{2}\right) + \cos\left(\frac{p\theta^2 r + q\theta^2 s}{2}\right) + \cos\left(\frac{p\theta^2 q - r\theta^2 s}{2}\right)\right]. \tag{3.79}$$

We have already observed how the usual rules of energy-momentum conservation still hold. The new observation here is that we have recovered the symmetry under any exchange of external momenta in spite of the fact that the phase associated to each vertex is only cyclically symmetric.

Now let us consider one-loop corrections. We have already discussed the emergence of infrared singularities in the two-point function connected to the IR/UV mixing. Here we want to investigate if the IR/UV mixing has similar implications for the 4-point vertex function. The relevant one-loop diagrams have the same structure of the corresponding commutative ones but as usual an appropriate θ-dependent phase factor is present in each vertex. The analysis proceeds using the same techniques already used in eqs. (3.60)-(3.65) and the final results is [34]

$$\Gamma^4(p,q,r,s) = -\frac{\delta^4(p+q+r+s)V_s(p,q,r,s)}{3\cdot 2^5\pi^2}\lambda\left\{2\ln(\frac{\Lambda^2}{m_R^2}) + \ln(\frac{1}{m_R^2 p\theta^2 p}) + \ln(\frac{1}{m_R^2 q\theta^2 q}) + \right.$$

$$+\ln(\frac{1}{m_R^2 r\theta^2 r}) + \ln(\frac{1}{m_R^2 s\theta^2 s}) + \ln(\frac{1}{m_R^2 (q+r)\theta^2 (q+r)}) + \tag{3.80}$$

$$\left. +\ln(\frac{1}{m_R^2 (q+s)\theta^2 (q+s)}) + \ln(\frac{1}{m_R^2 (s+r)\theta^2 (s+r)}) + ...\right\} \tag{3.81}$$

that is again divergent for vanishing external momenta (or vanishing noncommutative parameter θ). As one could guess from simple arguments, whereas for the two point function the divergences were quadratic, here we find only logarithmic infrared divergences. However again we observe the effect of the IR/UV mixing: the UV logarithmic divergences of the commutative theory become the IR divergences of the corresponding nonplanar diagrams. It is also worth noticing that in spite of these new infrared divergences the theory has been shown to be renormalizable up to two loops [80].

3.3 Unsolved problems for QFT in κ-Minkowski spacetime: $\lambda\varphi^4$ example

The issue of quantization of noncommutative theories on κ-Minkowski spacetime has not yet been extensively studied in literature (see however [66, 81, 82, 83, 84]). The obstacles to the implementation of these theories are closely connected to the properties of the $\stackrel{\kappa}{*}$ products. Here we want to briefly consider one example of these theories, $\lambda\varphi^4$ scalar theory, and discuss the basic differences with respect to the canonical counterparts. Construction of scalar field theory in functional formalism has been discussed in [66, 82]. The starting point is the generating functional

$$Z[J] = \int D\varphi \exp\left(i\int dx^4 \frac{1}{2}\partial_\mu\phi \stackrel{\kappa}{*} \partial^\mu\phi - \frac{1}{2}m^2\phi \stackrel{\kappa}{*} \phi + \lambda\phi \stackrel{\kappa}{*} \phi \stackrel{\kappa}{*} \phi \stackrel{\kappa}{*} \phi + \frac{1}{2}J \stackrel{\kappa}{*} \phi + \frac{1}{2}\phi \stackrel{\kappa}{*} J\right). \tag{3.82}$$

A first point to notice is that in the κ-Minkowski case some ambiguities arise already in the introduction of the sources. In this case in fact expressions like $\int dx^4 J \stackrel{\kappa}{*} \phi$ and $\int dx^4 \phi \stackrel{\kappa}{*} J$ do not give the same contributions. This is different from the canonical case where the corresponding terms give the same contributions thank to the properties of the canonical $\star$-product. The ambiguity between a $\int dx^4 J \stackrel{\kappa}{*} \phi$ and a $\int dx^4 \phi \stackrel{\kappa}{*} J$ source term has been tentatively approached, as shown in [66], by introducing both terms but some of the pathologies of QFT in κ-Minkowski might be even due to this initial assumption (which one may have to modify eventually).

Using relations (3.44-3.45) one can write the above expression in the momentum space as

$$Z[J] = \int D\varphi \exp i(S_0 + S_{int} + \frac{1}{2}\int dk^4 \mu(k_0)\left[J(k)\phi(\dot{-}k) + J(\dot{-}k)\phi(k)\right]), \tag{3.83}$$

where

$$\begin{aligned} S_0 &= \frac{1}{2}\int dk_1^4 dk_2^4 \delta^4(k_1 \dot{+} k_2)\phi(k_1)\left(\mathcal{C}_\kappa(k_2) - m^2\right)\phi(k_2) = \\ &= \frac{1}{2}\int dk^4 \mu(k_0)\phi(\dot{-}k)\left(\mathcal{C}_\kappa(k) - m^2\right)\phi(k), \end{aligned} \tag{3.84}$$

$$S_{int} = \lambda \int dk_1^4 ... dk_4^4 \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta^4(k_1 \dot{+} k_2 \dot{+} k_3 \dot{+} k_4). \tag{3.85}$$

One can also perform the Gaussian integration of (3.82) obtaining

$$Z_0[J] \equiv \exp\left(-\frac{i}{2}\int dk^4 \mu(k_0)\frac{J(k)J(\dot{-}k)}{\mathcal{C}_\kappa(k) - m^2}\right). \tag{3.86}$$

To obtain Green functions from the above expression one first needs a generalization of the functional derivative. A proper generalization results to be

$$\frac{\delta F[J]}{\delta J(k)} \equiv \lim_{\varepsilon\to 0} \frac{F[J(p) + \varepsilon\delta^4(p \dot{+} (\dot{-}k))] - F[J(p)]}{\varepsilon}, \tag{3.87}$$

$$\frac{\delta F[J]}{\delta J(\dot{-}k)} \equiv \lim_{\varepsilon\to 0} \frac{F[J(p) + \varepsilon\delta^4(p \dot{+} k)] - F[J(p)]}{\varepsilon}. \tag{3.88}$$

Using these new functional derivative, expression (3.44-3.45) and the fact that $\mathcal{C}_\kappa(k) = \mathcal{C}_\kappa(\dot{-}k)$, from (3.86) one can find the two-point function at the tree level

$$G(k, \dot{-}k') = \frac{i}{2}\mu(k_0)\mu(-k_0)\frac{\delta^4(p\dot{+}(\dot{-}p')) + \delta^4((\dot{-}p')\dot{+}p)}{\mathcal{C}_\kappa(k) - m^2}. \tag{3.89}$$

An important point to notice is that being $\delta^4(p\dot{+}(\dot{-}p')) = \delta^4((\dot{-}p')\dot{+}p)e^{-3p_0/\kappa} = \delta^4(p - p')e^{-3p_0/\kappa}$, expression (3.89) predicts the usual rule of energy momentum conservation in spite of the nontrivial coproducts governing the algebra of κ-Poincaré symmetries. So far, at the level of having considered only the tree level propagator, the theory still looks healthy. However serious pathologies are encountered already in the analysis of the one loop contribution to propagator and tree level vertex.

One-loop formulas for the propagator may be obtained [66, 82] with the usual procedure from (3.85) and (3.86). A distinction between planar and nonplanar diagrams, in analogy with the canonical case, results to be useful. For planar diagrams no problems arise and the energy-momentum conservations rules: they are the same as in the tree level formulas. Instead for planar diagrams nontrivial problems are encountered, mainly due to the fact that a modification of the momentum-conservation rule occurs that cannot even be described as a modified conservation law. The terms involving loop momenta in fact do not cancel each other in the argument of delta functions. With respect to the vertex function, functional formalism provide a way to overcome the problem of the ordering in the vertex, that has been considered one of the main obstacles towards the construction of a field theory. However other urgent problems in the vertex function occur already at the tree level. The most relevant problems is represented by the lacking of covariance under κ-Poincaré transformations of the argument of the δ^3 function which represents the energy-momentum-conservation law. These fundamental problems render non-reliable the construction of QFT on κ-Minkowski spacetime and however no general consensus is present in literature on the strategy adopted for quantization. These consideration led us to stop here our analysis of QFT theories in κ-Minkowski spacetime. From here on we will focus exclusively on QFT in canonical spacetime.

3.4 Gauge theories in canonical spacetime

In this section we want to sketch the construction of gauge theories in the case of canonical noncommutativity. Commutative non-abelian-gauge theories contain at most logarithmic divergences. So one might conjecture that noncommutative gauge theory to be free from quadratic and linear poles. We will see that this naive expectation is not fulfilled.

[3] More properly the argument of the δ is covariant in form but when it vanishes in a given inertial system it does not vanish in all other inertial systems. This has for example the illogical consequence that different inertial observers may not agree on the creation of a particle in a given process.

Noncommutative gauge theories are constructed as in the commutative case [85], starting from a Lie algebra whose generators satisfy the commutation rules

$$[T^a, T^b] = if^{abc}T^c, \tag{3.90}$$

where f^{abc} are the structure constants of the algebra. The transformations of the fields can be defined as

$$\delta_\alpha \psi(x) = i\alpha(x) \star \psi(x), \tag{3.91}$$

$$\delta_\alpha \bar{\psi}(x) = -i\psi(x) \star \alpha(x), \tag{3.92}$$

$$\delta_\alpha A_\mu = \partial_\mu \alpha(x) + i[\alpha(x), A_\mu]_\star,$$

where $[a, b]_\star = a \star b - b \star a$ has been already introduced in Section 3.1.2. and $\alpha(x) \equiv \alpha_a(x)T^a$.

One can also define the field strength as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\nu, A_\mu]_\star.$$

This field strength has the noteworthy property of transforming under infinitesimal gauge transformations according to the adjoint representation of the gauge group

$$\delta_\alpha F_{\mu\nu} = i[\alpha(x), F_{\mu\nu}]_\star. \tag{3.93}$$

We observe also that gauge symmetry is realized on the fields

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi(x) = \delta_{[\alpha,\beta]}\psi(x), \tag{3.94}$$

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)F_{\mu\nu} = \delta_{[\alpha,\beta]}F_{\mu\nu}. \tag{3.95}$$

The usual Yang-Mills Lagrangian density can be modified in

$$L = -\frac{1}{4\pi g^2} Tr[F_{\mu\nu} \star F^{\mu\nu}], \tag{3.96}$$

where Tr acts, as usual, on the gauge indices. We notice that (3.96) is not invariant under gauge transformations. In fact using (3.93) is easy to check that

$$\delta_\alpha L = -\frac{1}{4\pi g^2} Tr[\delta_\alpha F_{\mu\nu} \star F^{\mu\nu} + F^{\mu\nu} \star \delta_\alpha F_{\mu\nu}] = \tag{3.97}$$

$$= -\frac{i}{4\pi g^2} Tr[\alpha(x) \star F_{\mu\nu} \star F^{\mu\nu} - \alpha(x) \star F_{\mu\nu} \star F^{\mu\nu}] \neq 0. \tag{3.98}$$

However when we consider the action

$$S = -\frac{1}{4\pi g^2} \int dx^4 Tr[F_{\mu\nu} \star F^{\mu\nu}], \tag{3.99}$$

we recover the gauge invariance ($\delta_\alpha S = 0$). We observe that to recover gauge invariance the cyclicity property of the $\star$-product under integration is crucial[4]. The action (3.99) has all the

[4]In the κ-Minkowski case where the $\overset{\kappa}{*}$ product is not cyclically invariant, attempts of construction of gauge theories have been so far unsuccessful.

required features to represent a generalization of the $U(N)$ Yang-Mills action in a canonical noncommutative framework. We observe also that the procedure just outlined is of rather wide applicability in dealing with noncommutative spacetimes. The only property one needs is the mentioned cyclical invariance of the star product under integration. It is also worth noticing that, differently from the commutative case, (3.99) is written in terms of the U(N) gauge bosons in such a way that one cannot separate out a SU(N) sector from a residual U(1) sector[5].

One can also define covariant derivatives as

$$D_\mu \psi \equiv \partial_\mu \psi - i A_\mu \star \psi. \tag{3.100}$$

It is easy to check that they have the right gauge transformation properties

$$\delta_\alpha (D_\mu \psi) = i\alpha(x) \star D_\mu \psi. \tag{3.101}$$

With the notion of covariant derivative one can construct an action for the spinor fields

$$S = \int dx^4 \bar{\psi} \star (\gamma^\mu D_\mu - m)\psi, \tag{3.102}$$

whose gauge invariance is again easily verified though again its Lagrangian density is not gauge invariant.

To analyze the properties of noncommutative Yang-Mills theory, one needs to adapt the standard Faddeev-Popov technique to the noncommutative case. Here we will not discuss this technical point (see however [86, 87, 88]) since we are mainly interested in the phenomenon of the IR/UV connection, which is largely independent from this issue.

Feynman rules for canonical noncommutative gauge theory can be obtained from the actions (3.99) and (3.102). In particular explicit calculations of the gauge bosons self energy [35] lead to the result

$$\Pi^{\mu\nu}(p) = 8g^2(N_s + 2 - N_f)\alpha \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^4}, \tag{3.103}$$

where N_s is the number of scalar degrees of freedom and N_f is the number of fermionic degree of freedoms in the theory.

We will comment on the phenomenological implication of this result in the next chapter. Here we observe only that again the quadratic pole arises in the limit $p \to 0$ and in limit $\theta \to 0$ (again for $\theta \to 0$ one does not recover the commutative theory). We also observe that the coefficient of the quadratic divergent term is proportional to the number of bosonic degrees of freedom minus the number of fermionic degrees of freedom of the theory and that the gauge bosons contributes with two degrees of freedom (the contribution 2 in N_s+2-N_f). In particular if one has the same number of bosonic and fermionic degrees of freedom this coefficient vanishes. This occurs in supersymmetric theories and also in softly-broken SUSY theories. We will show

[5]The interaction terms couple these two sectors since in general $\det(A \star B) \neq \det(A) \star \det(B)$.

an example in the next chapter. Moreover it is worth noticing that while renormalizability of gauge theories has been largely studied (see e.g.[88, 89]) an all-order prove of renormalizability is still lacking.

3.5 Supersymmetric theories in canonical spacetime

There are two main motivations that render noncommutative supersymmetric theories interesting [90, 91, 92]. The first is that, as discussed in the case of the noncommutative gauge theories, an equal number of fermionic degrees of freedom and bosonic degrees of freedom may improve the infrared behavior of noncommutative theories. The second is the expectation that, as in the commutative case, SUSY noncommutative theories might manifest a more regular ultraviolet behavior. It will be not surprising to discover that actually both ultraviolet and infrared properties of noncommutative theories are improved by SUSY.

A first point to consider in the construction of a supersymmetric noncommutative theory is the compatibility of the commutation relations $[x_\mu, x_\nu] = i\theta_{\mu\nu}$ whit the supersymmetric algebra. Here we will consider only $\mathcal{N}$=1 supersymmetric theories although the general construction does not depend on the number of supersymmetries nor on the number of spacetime dimensions.

A superspace formulation of supersymmetry was given in [93] where instead of investigating the noncommutative superspace formalism it was considered the usual superspace and superfields. The result is that given the commutative supersymmetric action written in terms of superfields it is possible to obtain the noncommutative supersymmetric action by the only prescription of replacing the ordinary product between superfields with the $\star$-product.

We use the standard notation of [94], (except for the spacetime indices), which are denote here by μ, ν,... We start considering the chiral superfields which satisfy $\bar{D}_{\dot\alpha}\Phi = 0$. Using the coordinates $y^m = x^m + i\theta\sigma^m\bar\theta$, chiral superfields can be written as $\Phi(y, \theta, \bar\theta) = A(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)$. The supersymmetry transformations are identical to the ones of the commutative counterparts simply because we are considering the ordinary superfields

$$\begin{aligned}
&\delta_\xi A = \sqrt{2}\xi\psi, \\
&\delta_\xi \psi = i\sqrt{2}\sigma^m\bar\xi\partial_m A + \sqrt{2}\xi F, \\
&\delta_\xi F = i\sqrt{2}\bar\xi\bar\sigma^m\partial_m\psi.
\end{aligned} \tag{3.104}$$

The most generic action which can be constructed from the chiral superfields Φ^i takes the form[6]

$$S = \int d^4x \left(\int d^2\theta d^2\bar\theta K(\Phi^i, \Phi^{\dagger j})_\star + \left[\int d^2\theta W(\Phi^i)_\star + h.c. \right] \right), \tag{3.105}$$

[6] We use the notation $(\prod_{i=1}^n f_i)_\star = f_1 \star f_2 \star \cdots \star f_n$. Also notice that standard notation θ for both noncommutative parameters and SUSY Grassmann variables. From the context it should be clear when we refer to one or to the other.

where $\int d^2\theta\,\theta^2 = 1$ and $\int d^2\bar\theta\,\bar\theta^2 = 1$. This is invariant under $K(\Phi^i, \Phi^{+j})_\star \to K(\Phi^i, \Phi^{+j})_\star + F(\Phi)_\star + F(\Phi^+)^+_\star$. The action can be written in terms of the component fields straightforwardly, but rather than doing this in full generality we focus on a couple of specific examples. First we consider the action with $K = \Phi^+ \star \Phi + a\Phi \star \Phi \star (\Phi^+) + a^*\Phi \star (\Phi^+) \star (\Phi^+)$ and $W = 0$, where a is some numerical coefficient. Note that in this case the part of the action which depends on F becomes

$$S|_F = \int d^4x \left(F^+F + (aF(F \star A) + aF(A \star F^+) + aF(F \star A^+) + h.c)\right). \tag{3.106}$$

This action clearly contains the derivatives of the auxiliary field F. Thus F may become a propagating field if the noncommutative parameter $\theta^{0\mu} \neq 0$ for some μ. However in the case which are relevant in our future discussions the canonical Kähler potential is of the type $K = \sum_i \Phi_i^+ \star \Phi_i$ and in this case the action with non vanishing superpotential does not involve derivatives of F and then F plays the role of an auxiliary field, which can be eliminated as in the commutative case.

A model on which we want focusing our attention (and also the simplest supersymmetric model) is the Wess-Zumino model. Its action in terms of superfields reads

$$S_{WZ} = \int d^4x \left(\int d^2\theta d^2\bar\theta\, \Phi_i^+ \star \Phi_i + \left[\int d^2\theta \left(\frac{1}{2}m_{ij}\Phi_i\star\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\star\Phi_j\star\Phi_k + g_i\Phi_i\right) + h.c.\right]\right), \tag{3.107}$$

where the mass matrix m_{ij} is symmetric in its indices but the coupling g_{ijk} is not necessarily symmetric. One easily finds that

$$\begin{aligned} S_{WZ} =& \int d^4x \left(-\partial_\mu A_i^+ \partial^\mu A_i + i\partial_\mu \psi_i^+ \bar\sigma^\mu \psi_i + F_i^+ F_i\right) \\ &+ \int d^4x \left[\frac{1}{3}g_{ijk}\left(F_i\, A_j \star A_k + F_j\, A_k \star A_i + F_k\, A_i \star A_j - A_i\, \psi_j \star \psi_k - A_j\, \psi_k \star \psi_i - A_k\, \psi_i \star \psi_j\right)\right. \\ &\left. + g_i F_i + m_{ij}\left(A_i F_j - \frac{1}{2}\psi_i\psi_j\right) + h.c\right]. \end{aligned} \tag{3.108}$$

The equation of motions of F_i is

$$F_i^+ = g_i + m_{ij}A_j + \frac{1}{3}\left(g_{ijk} + g_{kij} + g_{jki}\right) A_j \star A_k, \tag{3.109}$$

and the supersymmetry transformation becomes (3.104) with this F_i. We note that the typical scalar potential has the form $A^+\star A^+\star A\star A$ and the notion of holomorphy is still valid at $\theta \neq 0$. One can also consider vector superfield $V = V^+$ and construct supersymmetric gauge theories [90]. Here we only state the main results. A first result is that the Wess-Zumino model is renormalizable to all orders of perturbation theory [36] and no signs of infrared poles are found in the two point effective action.

Instead $\mathcal{N}$=1 and $\mathcal{N}$=2 theories with generic $U(N)$ gauge group were found [37, 95, 96, 97, 98] to be divergent, at one loop, only in the two point function. However no quadratic divergences

were found. Only logarithmic divergences appear. UV divergences in the planar sector and IR divergences in the nonplanar sector have been found. They signal that UV/IR mixing is present in these theories though it has less strong effects. Supersymmetric noncommutative $\mathcal{N}$=4 theory was studied in [99, 100, 101]. In [101] it was shown to be free from infrared poles and, more remarkably, it was also argued to be finite (like its commutative counterpart).

3.6 Causality and unitarity in canonical spacetimes

In this section we briefly describe the issues of unitarity and causality in canonical noncommutative field theories. Perturbative unitarity was for the first time discussed in this context in [29]. It was noticed that unitarity is lost if noncommutativity involves the time coordinate. If M_{ab} is the transition matrix element between the state a and the state b, for on-shell matrix elements unitarity implies that

$$2\,\mathrm{Im}\, M_{ab} = \sum_n M_{an} M_{nb} \tag{3.110}$$

The sum over intermediate states is intended in the right hand side of the above expression. The rule (3.110) can be expressed in terms of Feynman graphs. This produces the so-called generalized-unitarity relations or cutting rules[7]. Cutting rules state that the imaginary part of a Feynman diagram can be obtained as follows: first one must cut the diagrams by a line through virtual lines, then one must place that virtual particle on-shell by replacing the propagator with a delta function

$$\frac{1}{p^2 - m^2 + i\varepsilon} \to -2\pi i \delta(p^2 - m^2), \tag{3.111}$$

wherever the cut intersects the virtual line, and finally the sum over all cuts is the imaginary part of the Feynman diagram. For example in the case of the two-point function in the noncommutative φ^3 theory one has that unitarity implies what reported in Fig.3.6 and explicit

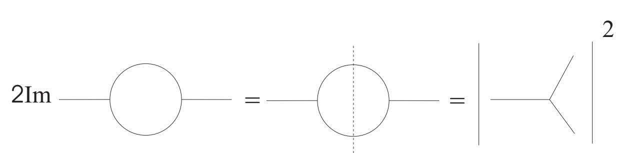

Figure 3.6: Cutting rule in the 2-point function for the φ^3 theory.

calculations shows [29, 31] that this relation is only satisfied for the case $\theta^{0i} = 0$. The same result has been proved for other graphs of the φ^4 scalar theory[8].

The analysis of causality leads to similar conclusions. It was noticed in [102] that nonlocal effects arising in NC field theories may lead to a violation of causality if the time coordinate

[7] Actually cutting rules are more restrictive than (3.110) since they involve off-shell conditions. Unitarity of the S-matrix (3.110) follows from the cutting rules.

[8] However for other studies that seem to disagree with these results see [32].

is involved in noncommutativity. The example treated in [102] is that of scattering of wave packets. An outgoing signal appears before the incoming particles reach each other. Acausal effects are not found in the case of space-space noncommutativity.

In general it seems that time-space noncommutativity leads to violation of both unitarity and causality whereas space-space noncommutativity preserves both of them[9].

3.7 Open problems related to the IR/UV mixing in canonical spacetime

So far we have discussed how low-energy poles appear in the Green functions of different NCQFT. We have also stressed how these poles originate from the ultraviolet sector of loop integrals so that they can be viewed as manifestations of the mixing between the UV and IR degrees of freedoms. It is also worth noticing that infrared poles are only one of the manifestations of mixing and theories that are free from these singularities must not be considered free from IR/UV mixing[10]. Moreover this correlation between short distances (IR) and large distances (UV) is not accidental in canonical noncommutative theories but, on the contrary, it is in a sense to be expected. This expectation comes directly from the commutation relations, which imply that

$$\Delta x_\mu \Delta x_\nu \gtrsim \theta. \tag{3.112}$$

This uncertainty relations implies that if a coordinate (say x) is known with an uncertainty $\Delta x \lesssim 1/\Lambda_0$ the other coordinate (say y) must be determined with an uncertainty $\Delta y \gtrsim \theta\Lambda_0$. This implies correlation between energies larger than Λ_0 with energies smaller than $1/\theta\Lambda_0$. Correlations of this type are rather unusual in the common language of physics where one usually observes the decoupling of the energy scales. Now we want to analyze more in detail the wide implications of this IR/UV mixing.

3.7.1 IR/UV mixing and renormalization group flow

We start our analysis of the implications of the IR/UV mixing from a discussion on the fate of the Wilsonian picture in QFT in a canonical spacetime. According to the usual Wilson picture to QFT, every theory, unless seen as fundamental, is defined with some cut-off which indicates our ignorance of the correct theory beyond the cut-off energy scale [104, 105] . Thus the theory is predictive at least below the cut-off, say Λ_0, and is understood as embedded in an unknown more fundamental theory beyond it. One typically studies how the theory appears at an observer which tests the theory at scales much lower than the cut-off. High-energy modes of the theory only generate the low-energy couplings of the effective theory. In the usual Wilsonian

[9] There are however some recent attempts to circumvent these problems (see for example [31, 103]).

[10] This point is often misunderstood in literature where absence of poles in the propagator if often identified with absence of the IR/UV mixing.

picture the high-energy cut-off, the bare couplings and the high-energy degrees of freedom may be entirely encoded in a definition of the parameters of the low-energy theory. Thus whatever a theory is at its natural high-energy scale its predictions for the low-energy regime depend only on a finite number of parameters. In this way different energy scales may be decoupled and one can perform low-energy experiments (i.e. "studying chemistry") and make predictions, knowing very little on the details of the high-energy interactions (i.e. the fundamental interactions). All the necessary information are encoded in a definition of the couplings (i.e. the fine structure constant).

Wilson-Polchinski renormalization group tells us how these couplings change upon varying the scale. Given an action S_{Λ_0} that describes quantum field theory up to the scale Λ_0 one can obtain the action S_{eff} which describes physics up to the scale $\Lambda < \Lambda_0$ by integrating out the degrees of freedom between Λ and Λ_0

$$Z_{\Lambda_0}[J] = \int D\phi_{\Lambda_0} \exp\left\{-S_{\Lambda_0}(\phi)\right\} = \tag{3.113}$$

$$= \int D\phi_{\Lambda} \int D\phi_{\Lambda\Lambda_0} \exp\left\{-S_{\Lambda_0}(\phi)\right\} = \int D\phi_{\Lambda} \exp\left\{-S_{eff}(\phi;\Lambda,\Lambda_0)\right\}, \tag{3.114}$$

where

$$\exp\left\{-S_{eff}(\phi;\Lambda,\Lambda_0)\right\} = \int D\phi_{\Lambda\Lambda_0} \exp\left\{-S(\phi)\right\}. \tag{3.115}$$

If one is interested in processes that take place at energy $E \ll \Lambda$ one can equivalently use S_{Λ_0} or $S_{eff}(\phi;\Lambda,\Lambda_0)$. The difference between the two possible choices lies in the fact that while using S_{Λ_0} one has to integrate in a huge range in the loop momenta (the range $0 - \Lambda_0$), in $S_{eff}(\phi;\Lambda,\Lambda_0)$ the integration in the range Λ-Λ_0 has already been performed and its effects are encoded in the coefficients of $S_{eff}(\phi;\Lambda,\Lambda_0)$. However the price to pay in this second case is that in general $S_{eff}(\phi;\Lambda,\Lambda_0)$ involves infinite interactions. From the fact that it must be

$$\partial_\Lambda Z_{\Lambda_0}[J] = 0, \tag{3.116}$$

one can also write the following integral-differential-flux equation (known as Polchinski equation) for the effective action

$$\Lambda\partial_\Lambda S_{eff} = \frac{1}{2}\int dp^4 \, (2\pi)^8 \, \Lambda\partial_\Lambda(D_\Lambda) \left\{ \frac{\delta S_{eff}}{\delta\phi(p)} \frac{\delta S_{eff}}{\delta\phi(-p)} - \frac{\delta^2 S_{eff}}{\delta\phi(p)\delta\phi(-p)} \right\} \tag{3.117}$$

with the initial condition

$$S_{eff}\left[\phi,\Lambda_0\right] = S\left[\phi,\Lambda_0\right], \tag{3.118}$$

where D_Λ is a cut-off function which is equal to one below Λ and rapidly vanishes above Λ .As for the action one might rewrite the same equation for the connected action $W[J]$ or the 1PI-effective action Γ[11]. Although in principle (3.117) might be valid at non-perturbative level (see

[11]It is worth observing that the equation for $W[J]$ can be regarded as an infinite-dimensional heat equation.

i.e. [106, 107]), one may use it for a perturbative calculation by performing a vertex expansion of the action

$$S_{eff}\,[\varphi,\Lambda,\Lambda_0] = \sum_{n=1}^{\infty} \frac{1}{(2n!)} \int_{p1...p_{2n}} C_{2n}(p_1,...,p_{2n},\Lambda,\Lambda_0)\,(2\pi)^4\,\delta^4(p_1+...+p_{2n})\varphi(p_1)...\varphi(p_{2n}) \quad , \tag{3.119}$$

where vertices C_{2n} may be evaluated by loop-expansion. The same may be done for the 1PI-effective action

$$\Gamma_{\Lambda,\Lambda_0}\,[\varphi] = \sum_{n=1}^{\infty} \frac{1}{(2n!)} \int_{p1...p_{2n}} \Gamma^{2n}_{\Lambda,\Lambda_0}(p_1,...,p_{2n})\,(2\pi)^4\,\delta^4(p_1+...+p_{2n})\varphi(p_1)...\varphi(p_{2n}). \tag{3.120}$$

One obtains for the vertices an equation of the form

$$\Lambda\partial_\Lambda\Gamma^{2n}_{\Lambda,\Lambda_0} = \mathcal{F}[\Gamma^2_{\Lambda,\Lambda_0},...,\Gamma^{2n+2}_{\Lambda,\Lambda_0}] \tag{3.121}$$

with an appropriate function $\mathcal{F}$. Then one can isolate in each of $\Gamma^{2n}_{\Lambda,\Lambda_0}$ the contributions of the relevant operators and of the irrelevant operators. Relevant operators are those operators whose couplings increase along the renormalization-group flow whereas irrelevant operators are those whose coupling are suppressed in the flow. The couplings of the relevant operators are all one needs for a low-energy theory (the irrelevant-operator couplings vanish in the infrared). For example one can write

$$\Gamma^2_{\Lambda,\Lambda_0}(p) = \Gamma^2_{\Lambda,\Lambda_0}(p)\big|_{p=p_0} + \left.\frac{\partial\Gamma^2_{\Lambda,\Lambda_0}(p)}{\partial p^2}\right|_{p=p_0}(p^2-p_0^2) + \Gamma^{2\ irr}_{\Lambda,\Lambda_0}(p) \tag{3.122}$$

$$\Gamma^4_{\Lambda,\Lambda_0}(p) = \Gamma^4_{\Lambda,\Lambda_0}(p)\big|_{p=p_0} + \Gamma^{4\ irr}_{\Lambda,\Lambda_0}(p), \tag{3.123}$$

where p_0 is the renormalization point (e.g. the scale at which renormalized coupling are fixed) and $\Gamma^{2\ irr}_{\Lambda,\Lambda_0}(p)$, $\Gamma^{4\ irr}_{\Lambda,\Lambda_0}(p)$ represent the irrelevant contributions to the two-point and four-point effective actions.

The same formal techniques can be applied to the canonical-noncommutative framework but some important differences emerge. In the commutative case, at any order of perturbation theory, it can be shown that irrelevant operators $\Gamma^{n,irr}_{\Lambda,\Lambda_0}(p)$ are suppressed by positive powers of Λ/Λ_0. This means that the low-energy theory $\Lambda << \Lambda_0$ depends on the high-energy theory only through the renormalized mass m_R and the renormalized coupling λ_R (up to corrections of order Λ/Λ_0). This is a finite number of parameters. The theory is renormalizable and the energy-scale decoupling mechanism works.

In the noncommutative case one finds [108] that as long as $p \ll \Lambda_0$ and $p_0, p \gg 1/\theta\Lambda_0$, $\Gamma^{2\ irr}_{\Lambda,\Lambda_0}$, for example, depends on Λ_0 in a exponentially suppressed way so that Wilsonian picture still holds: there is negligible influence on the physics at momentum scales p from the high-energy (Λ_0) sector of the theory. In these regime, energy-scale decoupling still works. Instead if one

considers external momenta p less than $1/\theta\Lambda_0$,one finds

$$\Gamma^{2\ irr}_{\Lambda,\Lambda_0}(p) \simeq \frac{\lambda}{96\pi^2}\Lambda_0^2 + ..., \tag{3.124}$$

which means that the Wilsonian picture of energy-scale decoupling is spoiled and that low-energy prediction under the scale $1/\theta\Lambda_0$ are highly sensitive to the (unknown) details of the ultraviolet sector of the theory[12]. This spoils the usefulness of the concept of effective low-energy action useless and affects the procedure usually adopted to test a physical model by comparing the predictions of the model with the low-energy data. We will investigate this important point in the next chapter.

3.7.2 IR/UV mixing and the subtraction point

Other effects related to the IR/UV mixing are the different scaling laws of the Green functions at different momenta and certain problems with the choice of the subtraction point. Let us consider as an example the case of the scalar theory already analyzed. For the two-point function we have for large Λ the scaling

$$\Gamma^2(\mu) \simeq \frac{\lambda}{48\pi^2}\Lambda^2, \tag{3.125}$$

if $\mu \neq 0$. Instead, if $\mu = 0$, we have the scaling

$$\Gamma^2(\mu) \simeq \frac{\lambda}{32\pi^2}\Lambda^2. \tag{3.126}$$

Closely related to this difference in the scaling of the Green function is the problem of the choice of the renormalization point. If we set renormalization conditions at a momentum scale $\mu \neq 0$ we find for the one-loop renormalized parameters

$$m_R^2 = m^2 + \frac{\lambda}{8\pi^2}\left[\Lambda^2 - m^2 \ln\frac{\Lambda^2}{m^2}\right], \tag{3.127}$$

$$\lambda_R = \lambda - \frac{\lambda}{8\pi^2}\ln\frac{\Lambda^2}{m^2}. \tag{3.128}$$

Instead if we choose a subtraction point $\mu = 0$ the renormalized parameters are the same as in the commutative case and they read

$$\begin{aligned} m_R^2 &= m^2 + \frac{3\lambda}{16\pi^2}\left[\Lambda^2 - m^2 \ln\frac{\Lambda^2}{m^2}\right], \\ \lambda_R &= \lambda - \frac{9\lambda}{16\pi^2}\ln\frac{\Lambda^2}{m^2}. \end{aligned} \tag{3.129}$$

[12] It is important to notice that this does not imply the theory is nonrenormalizable. One can still formally consider the infinite cut-off limit and obtain predictions in terms of a finite number of parameters (this is what we mean with renormalizability). However physically, the infinite cut-off limit is only justified by the mechanism of energy-scale decoupling and therefore it is unmotivated in these theories in canonical noncommutative spacetime. In the next chapter we consider a theory with a large mass scale and find that, after removal of the cut-off scale, this large mass scale still affects significantly the low-energy sector of the theory.

This implies that if we choose $\mu = 0$ as renormalization point, (subtracted) Green functions for general external momenta are not finite in the $\Lambda \to \infty$ limit and the theory will appear to be non renormalizable. If we choose a subtraction point at $\mu \neq 0$, (subtracted) Green functions are finite but poles, of the type of the ones already described, appear for $\mu \to 0$.

Another way to obtain the renormalized parameters (3.129) is the one of considering the effective potential which is the generator of 1PI-Green functions at zero momentum

$$V_{eff} = \sum_{i=1}^{\infty} \frac{1}{n!} \Gamma^n(0, ..., 0) \varphi^n. \tag{3.130}$$

At one-loop level the effective potential is the same as in the noncommutative case since at zero momentum nonplanar diagrams give the same contributions of the planar ones:

$$V_{eff} = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi^4 + \frac{1}{2} \int^{\Lambda} \frac{dk^4}{(2\pi)^4} \ln(1 + \frac{3\lambda\varphi^2}{k^2 + m^2}). \tag{3.131}$$

The renormalized parameters are now obtained from the relations

$$\left. \frac{d^2 V_{eff}}{d\varphi^2} \right|_{\varphi=0} = m_R^2, \tag{3.132}$$

$$\left. \frac{d^4 V_{eff}}{d\varphi^4} \right|_{\varphi=0} = 6\lambda_R. \tag{3.133}$$

These relations will lead to a couple of equations identical to (3.129) which correspond to nonrenormalizable Green functions.

We also notice that the considerations we are doing hold rather in general and are not restricted to the scalar-theory case. For example similar problems manifest in the analysis of the noncommutative Gross-Neveu model [109]. In general if the renormalization conditions are set at zero external momentum the theory does not renormalize whereas if the renormalization conditions are set at a nonzero external momentum the theory renormalizes. The origin of this behaviors is in a sort of non analytic structure that canonical noncommutativity induces at zero momentum. Since the noncommutativity parameters appear in diagrams only through factor of the type $\exp(ip\theta k)$, for $p = 0$ the diagrams reproduce the commutative spacetime limit, whereas as soon as $p \neq 0$ the θ parameter cannot be ignored and induce large contribution at low energy. As we shall show in Chapter 5 this peculiarities of the zero-momentum limit have profound implications for nonperturbative estimates of the effective potential.

3.7.3 IR/UV Mixing and the Goldstone theorem.

In this section we discuss the implication of the UV/IR mixing for the validity of the Goldstone theorem which is at the basis of the mechanism of mass generation (Higgs mechanism). The statement of the Goldstone theorem roughly is that for every spontaneously broken symmetry (i.e. a symmetry of the action that is not a symmetry of the ground state of the theory)

there must be a massless particle. For example in the case of the linear sigma model with $O(N)$ symmetry broken to $O(N-1)$, the numbers of symmetries changes from $N(N-1)/2$ to $(N-1)(N-2)/2$ so that $N-1$ symmetries are broken and $N-1$ massless particles (Goldstone bosons) appear. At the quantum level instead of considering the action S one has to consider the effective action Γ which besides having has the same symmetries of the classical theory[13]

The problem of the validity of the Goldstone theorem in noncommutative theories was first addressed in [110]. We consider scalar $O(N)$ theory whose action is

$$S=\frac{1}{2}\left(\partial_\mu\phi^i\right)^2+\frac{1}{2}\mu^2\left(\phi^i\right)^2-\frac{\lambda}{4}\phi^i\star\phi^i\star\phi^i\star\phi^i, \tag{3.134}$$

where the sum over the i index is omitted.

This model enjoys the symmetry $\phi^i \rightarrow R^{ij}\phi^j$ where R is a spacetime constant, $N\times N$ orthogonal matrix $RR^T=I$. If $\mu^2>0$ the classical potential

$$V_{cl}(\phi)=\frac{1}{2}\mu^2\left(\phi^i\right)^2-\frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2, \tag{3.135}$$

has a minimum for the constant field configuration

$$(\phi_0^i)^2=\frac{\mu^2}{\lambda}. \tag{3.136}$$

Therefore the $O(N)$ symmetry of the action is no more a symmetry of the vacuum, which is to say that the $O(N)$ symmetry is broken. The relation (3.136) identifies the $(N-1)$ dimensional manifold (actually, in this case, a sphere) on which the classical potential assumes its minimum. We can choose one point on this manifold to identify with the vacuum of the broken phase. We choose the configuration

$$\phi_0=(0,...,0,v)\,, \tag{3.137}$$

where $v\equiv\dfrac{\mu}{\sqrt{\lambda}}$. Hence we define the new fields

$$\begin{cases}\pi_i\equiv\phi^i\\ \sigma\equiv\phi^n-v\end{cases} \tag{3.138}$$

in such a way that $\langle\sigma\rangle=0$. The action (3.134) in terms of the new fields reads

$$S=\frac{1}{2}\left(\partial_\mu\pi^k\right)^2+\frac{1}{2}\left(\partial_\mu\sigma\right)^2+\frac{1}{2}(2\mu^2)\sigma^2+\frac{\lambda}{2}\left[\left(\pi^k\right)^2\right]^2+\frac{\lambda}{2}\sigma^4+\lambda v\sigma\left(\pi^k\right)^2+\frac{\lambda}{2}\sigma^2\left(\pi^k\right)^2+\lambda v\sigma^3. \tag{3.139}$$

The absence of the terms quadratic in the pion filed π^k is a prove of the Goldstone theorem at the tree level in canonical noncommutative theories. Also the breaking of the symmetry from $O(N)$ down to $O(N-1)$ is manifest. To check if the Goldstone theorem also holds at the quantum level one must verify that pion fields remain massless at the quantum level as well.

[13]If regularization does not break these symmetries.

This is the case if $\Gamma^2_{\pi\pi}(p)$ vanish at $p = 0$. Explicit one-loop calculation of $\Gamma^2_{\pi\pi}(p)$ have been carried out in [110]. The result is that if one considers first the $\Lambda \to \infty$ limit and then the $p \to 0$ limit $\Gamma^2_{\pi\pi}(p)$ does not vanish, actually it diverges. Instead if one first imposes $p = 0$ and then considers the $\Lambda \to \infty$ limit one recovers $\Gamma^2_{\pi\pi}(0) = 0$ and the validity of the Goldstone theorem[14]. This is another manifestation of the failure of the commutation of the zero-momentum limit ($p \to 0$) with large cut-off limit ($\Lambda \to \infty$), again a consequence of the IR/UV Mixing.

3.7.4 IR/UV Mixing and the scalar-theory phase diagram

We have discussed various problems connected to the zero-momentum limit of these theories. For example we have seen that Green functions at $p \neq 0$ are not renormalized by renormalization conditions fixed at zero-momentum, whereas, if the renormalization conditions are set away from $p = 0$, the Green functions exhibit a pole in the zero-momentum limit after removal of the cut-off ($\Lambda \to \infty$). The stiffness of the zero-momentum modes ($\Gamma^2(p) \stackrel{p\to 0}{\to} \infty$) has direct, and deep, implications also for the analysis of phase transitions. We know in fact that phase transitions are related to the condensations of some momentum modes and, in particular, phase-transition to translation-invariant vacuum are related to the condensation of the zero-momentum modes. We want to observe how in canonical noncommutative theories just because zero-momentum modes are stiff, transition to translation-invariant ordered phases are not trivial. As an example we consider the Ward identities for the scalar theory with global $O(2)$ symmetry [113]. The action of this theory is

$$S = -\frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 + \frac{\lambda}{4}\phi^i \star \phi^i \star \phi^i \star \phi^i. \tag{3.140}$$

In the symmetric phase the Ward identities are

$$\begin{aligned} \left.\frac{\delta^2\Gamma}{\delta\phi_1^2}\right|_{\phi_1=\phi_2=0} &= \left.\frac{\delta^2\Gamma}{\delta\phi_2^2}\right|_{\phi_1=\phi_2=0}, \\ \left.\frac{\delta^4\Gamma}{\delta\phi_1^4}\right|_{\phi_1=\phi_2=0} &= 3\left.\frac{\delta^4\Gamma}{\delta\phi_1^2\delta\phi_2^2}\right|_{\phi_1=\phi_2=0}. \end{aligned} \tag{3.141}$$

In the broken phase the symmetric vacuum becomes unstable and one, as usual, finds the new vacuum by a shift of the fields

$$\begin{aligned} \phi_1 &= \sigma + v, \\ \phi_2 &= \pi. \end{aligned} \tag{3.142}$$

Ward identities in the broken phase now read

$$v\left.\frac{\delta^2\Gamma}{\delta\pi^2}\right|_{\sigma=\pi=0} = \left.\frac{\delta\Gamma}{\delta\sigma}\right|_{\sigma=\pi=0}. \tag{3.143}$$

[14] It is worth observing that absence of violations of the Goldstone theorem has been proved at one loop in noncommutative-scalar U(N) theories in [111] or in other particular cases [112].

We know that in the corresponding theory in commutative spacetime both (3.141) and (3.143) hold true. In the noncommutative theory explicit one-loop calculations [113] show that the identities of the symmetric case (3.141) are still satisfied but the identities of the translational-invariant broken phase (3.143) are violated. The point is that the shift $\phi_1 = \sigma + v$ implicitly assumes a translational-invariant vacuum. If one considers transitions to a vacuum $v(x)$ which is not translational invariant one obtains the following Ward identities

$$\int \frac{dp^4}{(2\pi)^4} v(-p) \left. \frac{\delta^2\Gamma}{\delta\pi(p_1)\pi(p)} \right|_{\sigma=\pi=0} = \left. \frac{\delta\Gamma}{\delta\sigma(p_1)} \right|_{\sigma=\pi=0}, \tag{3.144}$$

which have been shown to be verified at one loop [113, 114]. This argument strongly favors the idea of stable nonuniform phases.

Phase transitions in scalar theories have been more carefully analyzed in Ref.[115] using a self-consistent one-loop analysis. The authors find some evidence of condensation of nonzero modes corresponding to an ordered phase which breaks translational invariance. Also relying on the natural assumption that at fixed cut-off Λ, in the $\theta \to 0$ limit one must recover the ordered translational-invariant phase of the commutative theory, Ref.[115] the phase diagrams here reported in for the scalar $\lambda\phi^4$ theory

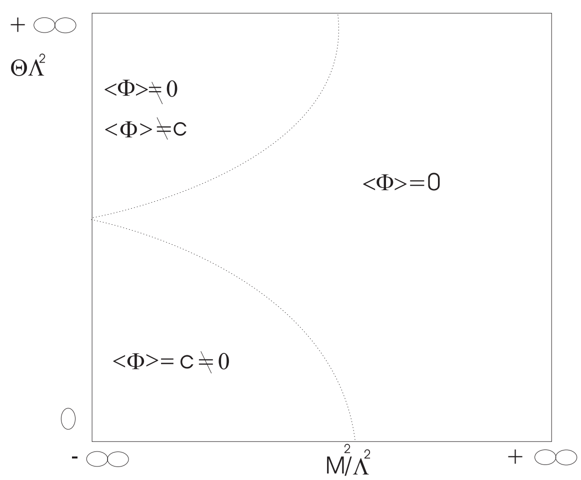

Figure 3.7: Phase diagram of the noncommutative $\lambda\varphi^4$-theory in the $(m^2/\Lambda^2,\, \theta\Lambda^2)$ plane.

Essentially three phases can be distinguished. The first phase is the ordered phase, characterized by the zero vacuum expectation value $\langle\phi\rangle = 0$. This phase is dominant for large positive values of the ratio m^2/Λ^2. For sufficiently large but negative values of the ratio m^2/Λ^2 one encounters the ordered phase $\langle\phi\rangle \neq 0$. For small value of $\theta\Lambda^2$ one has a translational-invariant ordered phase characterized by $\langle\phi\rangle = c$ similar to the one found in the commutative limit. For large value of $\theta\Lambda^2$ (i.e. in the limit in which only planar diagrams contribute) one finds an ordered translational-non-invariant phase which has been argued [115] to be a stripe phase,

characterized by a vev of the type $\langle\phi\rangle = A\cos(p_c x)$. Where p_c is the condensating momentum. In Chapter 5 we explore these issues concerning phase transitions in canonical noncommutative spacetimes using the Corwall-Jackiw-Tomboulis approach, a powerful nonperturbative technique of evaluation of the effective potential. This will also allow us to investigate the implications for nonperturbative renormalizability of the peculiar structure of the zero-momentum limit of theories in canonical noncommutative spacetime which was here briefly introduced in Subsection 3.7.

Chapter 4

Critical analysis of the phenomenology in CNC spacetimes

* In this chapter we investigate the implications of the IR/UV mixing for the derivation of experimental limits on the parameters of canonical noncommutative spacetimes. By analyzing a simple Wess-Zumino model in canonical noncommutative spacetime with soft supersymmetry breaking we explore the implications of ultraviolet supersymmetry on low-energy phenomenology. The fact that new physics in the ultraviolet can modify low-energy predictions affects significantly the derivation of limits on the noncommutativity parameters based on low-energy data. These are, in an appropriate sense here discussed, "conditional limits". We also find that some standard techniques for an effective low-energy description of theories with non-locality at short distance scales are only applicable in a regime where theories in canonical noncommutative spacetime lack any predictivity, because of the strong sensitivity to unknown UV physics.

4.1 IR/UV Mixing and Phenomenology in canonical spacetime

We have discussed how a key characteristic of field theories on canonical spacetimes, which originates from the commutation rules, is nonlocality. At least in the case of space/space noncommutativity ($\theta_{0i} = 0$), to which we limit our analysis for simplicity, this nonlocality is still tractable although it induces a characteristic mixing of the ultraviolet and infrared sectors of the theory. This IR/UV mixing has wide implications, including the possible emergence of infrared (zero-momentum) poles in the one-loop two-point functions. In particular one finds a quadratic pole for some integer-spin particles in non-SUSY theories [34], while in SUSY theories the poles, if at all present, are logarithmic [35, 36, 37]. It is noteworthy that these infrared singularities are introduced by loop corrections and originate from the ultraviolet part of the loop integration: at tree level the two-point functions are unmodified, but loop corrections involve the interaction vertices, which are modified already at tree level.

[0]* In this Chapter we discuss in detail the analysis reported more briefly in Ref.[47].

There has been considerable work attempting to set limits on the noncommutativity parameters θ by exploiting the modifications of the interaction vertices [38, 116, 40, 41] and the modifications of the dressed/full propagators [26]. Most of these analyses rely on our readily available low-energy data. The comparison between theoretical predictions and experimental data is usually done using a standard strategy (the methods of analysis which have served us well in the study of conventional theories in commutative spacetime). We are here mainly interested in understanding whether one should take into account some of the implications of the IR/UV mixing also at the level of the techniques by which one compares theoretical predictions with data. In Ref. [26] it was argued that the way in which low-energy data can be used to constrain the noncommutativity parameters is affected by the IR/UV mixing. These limits on the entries of the θ matrix might not have the usual interpretation: they could be seen only as "conditional limits", conditioned by the assumption that no contributions relevant for the analysis are induced by the ultraviolet. The study we report here is relevant for this delicate issue. By analyzing a simple noncommutative Wess-Zumino-type model, with soft supersymmetry breaking, we explore the implications of ultraviolet supersymmetry on low-energy phenomenology. Based on this analysis, and on the intuition it provides about other possible features of ultraviolet physics, we provide a characterization of low-energy limits on the noncommutativity parameters. Our analysis provides additional encouragement for combining, as proposed in Ref. [26], high-energy data, from astrophysics, with the more readily available low-energy data.

4.2 Effects of UV SUSY on IR physics

In this section we analyze a mass deformed Wess-Zumino model in canonical noncommutative spacetime. We emphasize the role that the UV scale of SUSY restoration plays in the IR sector of the model, and we also provide some more general remarks on the IR/UV mixing. This analysis will provide material for one of the points we raise in the later part of the paper, which concerns the nature of the bounds that can be set on the noncommutativity parameters using low-energy data.

4.2.1 A model with SUSY restoration in the UV

For definiteness, we present our observations, which have rather wide applicability, in the specific context of a mass deformed Wess-Zumino model, with action

$$S_{dwz} = S_0 + S_m + S_g, \tag{4.1}$$

$$S_0 = \int dx^4 \left\{ \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 + \frac{1}{2}\overline{\psi} i\partial\!\!\!/\psi \right\}, \tag{4.2}$$

$$S_m = \int dx^4 \left\{ \frac{1}{2}F^2 + \frac{1}{2}G^2 + m_s F\varphi_1 + m_s G\varphi_2 - \frac{1}{2}m_f\overline{\psi}\psi \right\}, \tag{4.3}$$

$$S_g = \int dx^4 g \left\{ F\star\varphi_1\star\varphi_1 - F\star\varphi_2\star\varphi_2 + G\star\varphi_1\star\varphi_2 + \right. \tag{4.4}$$
$$\left. + G\star\varphi_2\star\varphi_1 - \overline{\psi}\star\psi\star\varphi_1 - \overline{\psi}\star i\gamma^5\psi\star\varphi_2 \right\} .$$

φ_1 and φ_2 are bosonic/scalar degrees of freedom, while ψ denotes fermionic spin-1/2 degrees of freedom. F and G are auxiliary fields. The model is exactly supersymmetric (SUSY) if $m_s = m_f$. We consider the case $m_s < m_f$ in which supersymmetry is only "restored" in the ultraviolet (UV), where both m_s and m_f are negligible with respect to the high momenta involved.

The free propagators are not modified by canonical noncommutativity:

$$\Delta_{m_s}(p) \equiv \Delta_{\varphi_1\varphi_1}(p) = \Delta_{\varphi_2\varphi_2}(p) = \frac{i}{p^2 - m_s^2 + i\varepsilon} , \qquad \Delta_{FF}(p) = \Delta_{GG}(p) = p^2\Delta_{\varphi_1\varphi_1}(p), \tag{4.5}$$

$$\Delta_{F\varphi_1}(p) = \Delta_{\varphi_1 F}(p) = \Delta_{\varphi_2 G}(p) = \Delta_{G\varphi_2}(p) = -m_s\Delta_{\varphi_1\varphi_1}(p) , \qquad S(p) = \frac{i}{p\!\!\!/ - m_f} .$$

The vertices acquire the familiar θ-dependent phases:

$$V_{[\overline{\psi}\psi\varphi_1]} = -ig\cos(p_1\widetilde{p}_2) , \quad V_{[\overline{\psi}\psi\varphi_2]} = -i\gamma^5 g\cos(p_1\widetilde{p}_2) , \tag{4.6}$$
$$V_{[F\varphi_1\varphi_1]} = ig\cos(p_1\widetilde{p}_2) , \quad V_{[F\varphi_2\varphi_1]} = -\,ig\cos(p_1\widetilde{p}_2) , \quad V_{[G\varphi_1\varphi_2]} = 2ig\cos(p_1\widetilde{p}_2) .$$

[Notice that, taking into account momentum conservation at vertices, the momenta $p_1$and p_2 can be attributed equivalently to any of the three particles involved in each of the vertices.]

4.2.2 Self-energies and IR singularities

Self-energies will play a key role in our observations. Using the NC Feynman rules the self-energies for fermions and scalars can be evaluated straightforwardly. The one loop self-energy of the scalar field receives contributions from five Feynman diagrams, leading to the result

$$-i\Sigma_{1loop}(p) = -g^2 \int \frac{d^4k}{(2\pi)^4} \left\{ \left(8k^2 + 8m_s^2\right)\Delta_{m_s}(p)\Delta_{m_s}(p+k) + \right. \tag{4.7}$$
$$\left. - \left(8k^2 + 8m_f^2 + 8p\cdot k\right)\Delta_{m_f}(p)\Delta_{m_f}(p+k) \right\} \cos^2(k\widetilde{p}). \tag{4.8}$$

This expression can be seen as the sum of three terms, and each of these terms is the sum of a planar and of a nonplanar part: $-i\Sigma_{1loop}(p) = I_1^P(p) + I_1^{NP}(p) + I_2^P(p) + I_2^{NP}(p) + I_3^P(p) + I_3^{NP}(p)$ with

$I_1^P(p) + I_1^{NP}(p) \equiv \frac{1}{2}g^2 \int \frac{dk^4}{(2\pi)^4} \frac{8k^2+8m_s^2}{(k^2-m_s^2)((k+p)^2-m_s^2)} + \frac{1}{2}g^2 \int \frac{dk^4}{(2\pi)^4} \cos(2p\widetilde{k}) \frac{8k^2+8m_s^2}{(k^2-m_s^2)((k+p)^2-m_s^2)};$

$I_2^P(p) + I_2^{NP}(p) \equiv -\frac{1}{2}g^2 \int \frac{dk^4}{(2\pi)^4} \frac{8k^2+8m_f^2}{\left(k^2-m_f^2\right)\left((k+p)^2-m_f^2\right)} - \frac{1}{2}g^2 \int \frac{dk^4}{(2\pi)^4} \cos(2p\widetilde{k}) \frac{8k^2+8m_f^2}{\left(k^2-m_f^2\right)\left((k+p)^2-m_f^2\right)};$

$I_3^P(p) + I_3^{NP}(p) \equiv -\frac{1}{2}g^2 \int \frac{dk^4}{(2\pi)^4} \frac{8p\cdot k}{\left(k^2-m_f^2\right)\left((k+p)^2-m_f^2\right)}. - \frac{1}{2}g^2 \int \frac{dk^4}{(2\pi)^4} \cos(2p\widetilde{k}) \frac{8p\cdot k}{\left(k^2-m_f^2\right)\left((k+p)^2-m_f^2\right)};$

The planar terms involve integrations which are already done ordinarily in field theory in commutative spacetime. Their contributions lead, as in the commutative case, to logarithmic mass and wavefunction renormalization. We are here mainly interested in $\Sigma(p)_{1loop}^{NP(E)}$, the sum of the nonplanar contributions, which we study in the euclidean region. One easily finds[1]

$$\Sigma_{1loop}^{NP(E)}(p) = I_{1E}^{NP}(p) + I_{2E}^{NP}(p) + I_{3E}^{NP}(p) \ , \tag{4.9}$$

where

$$\begin{aligned} I_{1E}^{NP}(p) = \frac{g^2}{2\,(2\pi)^2} \int_0^1 da \Big\{ & \left[8m_s^2 + 4p^2(1-a)(2a-1)\right] K_0(2\,|\widetilde{p}|\,\sqrt{m_s^2+p^2a(1-a)}) + \\ & -\frac{4}{|\widetilde{p}|}\sqrt{m_s^2+p^2a(1-a)}K_1(2\,|\widetilde{p}|\,\sqrt{m_s^2+p^2a(1-a)}) \Big\} , \end{aligned} \tag{4.10}$$

$$I_{2E}^{NP}(p) = -\left[I_{1E}^{NP}(p)\right]_{m_s \to m_f} , \tag{4.11}$$

$$I_{3E}^{NP}(p) = -\frac{4}{(2\pi)^2} p^2 \frac{g^2}{2} \int_0^1 db b K_0(2\,|\widetilde{p}|\,\sqrt{m_f^2 + p^2 b(1-b)}) \ . \tag{4.12}$$

In the case of exact SUSY, $m_s = m_f$, the contributions I_{1E}^{NP} and I_{2E}^{NP} cancel each other, so that $\Sigma_{1loop}^{NP(E)} = I_{3E}^{NP}$ and there are no IR divergencies [36, 35].

In the general case, $m_s \neq m_f$, IR divergencies are present. Their structure depends on the relative magnitude of the SUSY-restoration scale $\Lambda_{SUSY} \simeq m_f$ and the noncommutativity scale $M_{nc} = \frac{1}{\sqrt{|\theta|}}$ (where $|\theta|$ denotes generically a characteristic size of the elements of the matrix $\theta_{\mu\nu}$).

If $M_{nc} < m_f$ and $p \ll \frac{M_{nc}^2}{m_f}$ the non-planar part of the self energy is well approximated by

$$\begin{aligned} \Sigma_{1loop}^{NP(E)}(p) \simeq \frac{g^2}{(2\pi)^2} \int_0^1 da \Big\{ & 6m_f^2 \ln\left(2\,|\widetilde{p}|\,\sqrt{m_f^2 + p^2a(1-a)}\right) + \\ & - 6m_s^2 \ln\left(2\,|\widetilde{p}|\,\sqrt{m_s^2 + p^2a(1-a)}\right) + \\ & + 2p^2(1-a)(3a-1)\left[\ln\left(\sqrt{m_f^2+p^2a(1-a)}\right) - \ln\left(\sqrt{m_s^2+p^2a(1-a)}\right)\right] \\ & + \left(m_s^2 - m_f^2\right)\left[6\ln 2 - 6\gamma + 1\right] + \\ & + 2p^2 a\left[\ln\left(2\,|\widetilde{p}|\,\sqrt{m_f^2+p^2a(1-a)}\right) - (\ln 2 - \gamma)\right]\Big\} \ . \end{aligned} \tag{4.13}$$

[1] $K_0(x)$ and $K_1(x)$ are modified Bessel functions of the second kind.

[This approximation is also valid for all $p < M_{nc}$ if $M_{nc} > m_f$, but we are mainly interested here in the case $M_{nc} < m_f$ which allows us to explore the implications for low-energy phenomena of SUSY restoration above M_{nc}.]

If $M_{nc} < m_f$ and $\frac{M_{nc}^2}{mf} \ll p \ll M_{nc}$ the non-planar part of the self energy is well approximated by

$$\Sigma_{1loop}^{NP(E)}(p) \simeq \frac{g^2}{(2\pi)^2}\int_0^1 da \left\{ -\frac{1}{|\widetilde{p}|^2} + \right.$$
$$- \ln\left(2\,|\widetilde{p}|\sqrt{m_f^2 + p^2 a(1-a)}\right)\left[6m_s^2 + 2p^2(1-a)(3a-1)\right] +$$
$$\left. + m_s^2[6\ln 2 - 6\gamma + 1] + 2p^2(1-a)[a(3\ln 2 - 3\gamma + \frac{1}{2}) - (\ln 2 - \gamma)] \right\}. \tag{4.14}$$

As a result of contributions coming from the UV portion of loop integrals, we are finding that (for $m_s \neq m_f$) the model is affected by logarithmic IR singularities (4.13) if $\frac{M_{nc}^2}{mf} \gg p$, but as soon as momenta are greater than $\frac{M_{nc}^2}{m_f}$ the dependence of the self-energy on momentum turns into an inverse-square law (4.14). In the limit $m_f \to \infty$, the case in which there is absolutely no SUSY (not even in the UV), the inverse-square law takes over immediately and the theory is affected by quadratic IR singularities. The case of exact SUSY $m_f = m_s$ is free from IR singularities, but of no interest for physics (Nature clearly does not enjoy exact SUSY).

The IR/UV mixing manifests in two (obviously connected) ways which is worth distinguishing: (1) The UV portion of loop integrals is responsible for some IR singularities of the self-energies, (2) the low-energy structure of the model can depend on m_f even when m_f is much higher than the energy scales being probed. There is no IR/UV decoupling.

4.2.3 Further effects on the low-energy sector from UV physics

The implications of supersymmetry for the IR sector of canonical noncommutative spacetimes are very profound. In our illustrative model one finds that exact SUSY leads to absence of IR divergences, if SUSY is only present in the UV (UV restoration of SUSY) one finds soft, logarithmic, IR divergences, and total absence of SUSY ($m_f \to \infty$) leads to quadratic IR divergences. While the presence of SUSY in the UV is clearly an example of UV physics with particularly significant implications for the IR sector of canonical noncommutative spacetimes, from this example we must deduce that in general the loss of decoupling between UV and IR sectors can be very severe. Other features of the UV sector, which perhaps have not even yet been contemplated in the literature, might have similarly pervasive implications for the IR sector.

A particularly interesting scenario is the one in which supersymmetry is restored at some high scale (which in our illustrative model is m_f) and then at some even higher scale, possibly identified with the so-called "quantum-gravity scale", the theory predicts additional structures, which in turn, again, would affect the infrared. The example of quantum gravity is particularly

significant since we have no robust (experimentally supported) information on this realm of physics, so it represents an example of UV physics for which our intuition might easily fail, and as a consequence our intuition for its implications for the IR sector of a field theory in canonical noncommutative spacetime might also easily fail.

As a way to emphasize the sensitivity of the IR sector to such unknown UV physics, it is worth noting here some formulas that describe features of our illustrative model from the perspective of a theory with fixed cutoff scale Λ. For renormalizable field theories in commutative spacetime the presence of such a cutoff would be basically irrelevant: if the cutoff is much higher than all scales of interest it will negligibly affect all predictions and it can be uneventfully removed through the limit $\Lambda \to \infty$. Importantly, in a renormalizable field theory in commutative spacetime the limit $\Lambda \to \infty$ is uneventful independently of whether or not we have introduced in the theory all the correct UV degrees of freedom hosted by Nature: the low-energy physics is anyway independent of (decoupled from) the UV sector.

For field theories in canonical noncommutative spacetime the limit $\Lambda \to \infty$ is not at all trivial, meaning that the structures/degrees of freedom encountered along the limiting procedure can in principle affect significantly the low-energy physics. One can take the $\Lambda \to \infty$ limit in a physically meaningful way only under the assumption that one has complete knowledge of the full theory of Nature (something which of course we cannot even contemplate).

The sensitivity of the IR sector to unknown UV physics is well characterized by considering, for fixed cutoff scale Λ, the nonplanar contributions to the two point functions. For the two-point function we already considered previously one finds:

$$I_{1E}^{NP} = \frac{g^2}{2}\left\{\frac{1}{(2\pi)^2}\int_0^1 da\left[8m_s^2+4p^2(2a-1)(1-a)\right]K_0(2\sqrt{\widetilde{p}^2+\frac{1}{\Lambda^2}}\sqrt{m_s^2+p^2a(1-a)})+\right.$$
$$\left.+\frac{4}{\sqrt{\widetilde{p}^2+\frac{1}{\Lambda^2}}}\left[\frac{\widetilde{p}^2}{\widetilde{p}^2+\frac{1}{\Lambda^2}}-2\right]\sqrt{m_s^2+p^2a(1-a)}K_1(2\sqrt{\widetilde{p}^2+\frac{1}{\Lambda^2}}\sqrt{m_s^2+p^2a(1-a)})\right\} \tag{4.15}$$

$$I_{2E}^{NP} = -I_{1E}^{NP}(m_s \to m_f) \tag{4.16}$$

$$I_{3E}^{NP} = -\frac{4}{(2\pi)^2}p^2\frac{g^2}{2}\int_0^1 db bK_0(2\sqrt{\widetilde{p}^2+\frac{1}{\Lambda^2}}\sqrt{m_f^2+p^2b(1-b)}) \tag{4.17}$$

Note that nonplanar diagrams are cutoff by $\Lambda_{eff} = \frac{1}{\sqrt{\widetilde{p}^2+\frac{1}{\Lambda^2}}}$. The self-energy is insensitive to the value of Λ as long as the condition $|\widetilde{p}| \gg \frac{1}{\Lambda}$ is satisfied. But for $|\widetilde{p}| < \frac{1}{\Lambda}$ there is an explicit dependence[2] on Λ signaling that the infrared sector is sensitive to new physics in the UV.

[2]It is worth noticing that for fixed cutoff Λ and $|\widetilde{p}| < \frac{1}{\Lambda}$ the self-energy is essentially independent of the noncommutativity parameters. This is due to the fact that under those conditions the nonplanar contributions are completely negligible. This might encourage one to contemplate the possibility of a physical cutoff scale Λ, but it is important to notice that such a scale would be observer dependent since ordinary Lorentz transformations still govern the transformations between inertial observers in canonical noncommutative spacetime [117]. (In

4.3 Conditional bounds on noncommutativity parameters from low-energy data

The main point of our work is that the observations made in the previous Section have significant implications for the comparison of low-energy experimental data with a theory in canonical noncommutative spacetime.

It is useful to note here a brief description of the conventional technique that allows to use low-energy data to set absolute (unconditional!) limits on the parameters of theories in commutative spacetime:

- **1C.** Data are taken in experiments involving particles with energies/momenta from some lower (IR) limit, $\mathcal{S}_{min}$ (we of course do not have available probes with wavelength, *e.g.*, larger than the size of the Universe) up to an upper limit, $\mathcal{S}_{max}$, which naturally coincides with the highest energy scales attainable in our laboratory experiments (and, in appropriate cases, the energy scales involved in certain observations in astrophysics).

- **2C.** We then compare these experimental results obtained at energy/momentum scales within the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ to the corresponding predictions of the theory of interest. In deriving these predictions we sometimes formally appear to use the whole structure of the theory, all the way to infinite energy/momentum; however, in reality, because of the IR/UV decoupling that holds in (renormalizable) theories in commutative Minkowski spacetime, the theoretical prediction only depends on the IR structure of the theory, up to energy/momentum scales which are not much bigger than $\mathcal{S}_{max}$. (For example, degrees of freedom with masses of order, say, $10^5\mathcal{S}_{max}$ would anyway not affect the relevant predictions).

- **3C.** If the theoretical predictions obtained in this way do not agree with the observations performed in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ we then conclude that the theory in question is to be abandoned.

- **4C.** If the theoretical predictions obtained in this way agree with the observations performed in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ we then conclude that the theory in question provides a valid description of phenomena up to energy/momentum scales of order $\mathcal{S}_{max}$. Typically the predictions of the theory will depend on some free parameters and this parameter space will be constrained by the requirement of agreeing with the observations. Values of the parameters that do not belong to this allowed portion of the parameter space are definitely

other noncommutative spacetimes, where the action of boosts is deformed, a cutoff scale can be introduced in an observer-independent way [58, 117], but this is not the case of canonical noncommutative spacetimes.) We shall disregard this possibility; however, in theories that already identify a preferred class of inertial observers, such as theories in canonical noncommutative spacetimes, the possibility of an observer-dependent cutoff scale cannot [117] be automatically dismissed.

(unconditionally) excluded, since nothing that we could introduce in the ultraviolet could modify the low-energy predictions. In light of the fact that the structure of the theory above $\mathcal{S}_{max}$ did not play any true role in the derivation of the predictions, the successful comparison with $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ experiments provides no particular encouragement for what concerns the validity of the theory at scales much above $\mathcal{S}_{max}$.

- **5C.** With precision measurements in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ we can sometimes put limits on features of the theory also slightly (up to a few orders of magnitude) above $\mathcal{S}_{max}$. For example, one of the parameters of the theory could be the mass of a certain particle and the contributions to low-energy processes due to that particle, while suppressed by its mass, can be tested in high-precision measurements.

For theories in canonical noncommutative spacetime the situation is quite different, as one infers from the analysis reported in the previous Section. The comparison between the theory and data taken in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ is much more delicate:

- **2NC.** From the observations made in the previous Section it follows that in a canonical noncommutative spacetime a truly reliable derivation of the predictions for the energy/momentum range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ requires full knowledge of the theory at all energy/momentum scales up to $M_{nc}^2/\mathcal{S}_{min}$ (and of course, if $M_{nc} \gg \mathcal{S}_{max}$, the scale $M_{nc}^2/\mathcal{S}_{min}$ can be much higher than both M_{nc} and $\mathcal{S}_{max}$). In particular, the IR/UV mixing is such that degrees of freedom with masses that are much above $\mathcal{S}_{max}$ still affect significantly the predictions of the theory in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$.

- **3NC.** So the theory can only be taken as a full description of Nature. It cannot be intended to give the right predictions only in some low-energy limit. If the predictions of such a theory are found to be in conflict with observations, it might still well be that the theory contains the right low-energy degrees of freedom, and that the disagreement is due to having adopted the wrong UV sector. So, from our more conventional perspective (in which we try to identify theories that contain the right degrees of freedom up to a certain scale) disagreement with observations does not force us to abandon the theory: it only invites us to introduce appropriate new physics in the UV sector.

- **4NC.** Similarly, if the theoretical predictions are found to agree with the observations performed in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ when some free parameters fall within a certain allowed portion of parameter space, values of the parameters that do not belong to that region of the parameter space cannot be conclusively excluded. They are excluded only **conditionally**, in the sense that their exclusion is only tentative, pending further exploration of the UV sector. Think for example of the illustrative model we considered in the preceding

Section. The $m_f \to \infty$ of that model is a model without any SUSY (not even in the UV sector). One could propose such a non-SUSY model and compare it to data obtained in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$. Clearly the need to agree with observations would then impose a severe (lower) bound on the noncommutativity scale, a key parameter of the theory, in order to suppress the IR divergences (*e.g.* effectively relegating those divergences at scales below $\mathcal{S}_{min}$). However, this bound on the noncommutativity scale would be only conditional, in the sense that modifying the theory only in the ultraviolet (*i.e.* where we would say it has not been tested with our data in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$) may be sufficient to lift the bound. In fact, SUSY in the ultraviolet sector (m_f large but finite) significantly softens the divergences used to set the bound. Whereas in commutative spacetime the bounds on parameter space apply directly to the structure of the theory in the range of energy/momentum scales that have been probed experimentally, in canonical noncommutative spacetime the information gained experimentally in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ leaves open two possibilities: it may still, as in the case of theories in commutative spacetime, constrain the parameters of the theory in that same range of energy/momentum scales, but one cannot exclude the possibility that our low-energy observations are instead primarily a manifestation of some features of the UV sector (transferred to the low-energy sector via the IR/UV mixing) and therefore cannot be used to constrain the low-energy structure of the theory. If there is disagreement between theory and experiments in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ one would normally assume that some aspects (*e.g.* the field content) of the theory must be changed in that same range of energy/momentum scales, instead in canonical noncommutative spacetime that same disagreement could be solved not only by introducing new features in the $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ region but also by introducing new features in the UV sector of the theory.

- **5NC.** Since data taken in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ do not even give definitive information on the structure of the theory in that same range, it is of course true that measurements in the range $\{\mathcal{S}_{min}, \mathcal{S}_{max}\}$ cannot be used to put limits on features of the theory even just slightly above $\mathcal{S}_{max}$, no matter how precise those measurements are. However, just because features of the UV sector affect the low-energy physics, under the assumption that the spacetime is indeed canonically noncommutative, one can gain insight of the UV structure of the theory, even just using low-energy data. For example, some of the observations made in the previous Section provide an opportunity to discover UV SUSY even just using low-energy data: if data allowed us to identify an energy/momentum scale at which the self-energy changed its qualitative dependence on momentum in the way described by comparison of Eqs. (4.13) and (4.14), we could then infer rather robustly the presence of SUSY at high energies and (if the value of the noncommutativity scale was

deduced from some other observations) we could even deduce the scale of SUSY restoration.

In summary we found that the predictions of a canonical noncommutative theory in the low-energy (i.e. experimentally accessible) sector of theory depend strongly not only on the low-energy structure of the theory but also on its high-energy structure. This is different from the case of commutative theories, where low-energy predictions are independent of the high-energy degrees of freedom. The phenomenological implications of this lack of energy-scale decoupling are of course very striking. To reliably falsify or accept a theory with low energy data it is not enough to specify the low-energy sector of the theory one is considering, but one must also fully specify the high-energy sector. Two theories with the same low-energy sector but different high energy sectors may require different parameter values to fit the data.

4.4 Futility of approaches based on expansion in powers of θ

The observations reported in the preceding section indicate that some of the standard techniques used in phenomenology require a prudent implementation in the context of theories in canonical noncommutative spacetimes. We want to emphasize in this section that for one of the techniques which served us well in the analysis of theories in commutative spacetime there are even more severe limitations to the applicability in the context of theories in canonical noncommutative spacetimes. This is the technique that relies on the truncation of a power series in one of the parameters of the theory: we argue that, at the quantum-field-theory level, the results obtained by truncating a power series in θ do not provide a reliable approximation of the full theory. This type of truncation, which has been widely used in the literature [85, 118, 119, 120, 121, 122, 123]), is based on the inclusion of only a few terms in the θ-expansion of the Moyal $\star$-product. For example up to the second order in θ one could write

$$\begin{aligned}\varphi_1(x)\star\varphi_2(x) = \varphi_1(x)\varphi_2(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu\varphi_1(x)\partial_\nu\varphi_2(x) + \\ -\frac{1}{8}\theta^{\alpha\beta}\theta^{\mu\nu}\partial_\alpha\partial_\mu\varphi_1(x)\partial_\beta\partial_\nu\varphi_2(x) + O(\theta^3)\end{aligned} \tag{4.18}$$

The resulting action constructed with the truncated $\star$-product (4.18) depends only on a finite number of derivatives so it is local, unlike the full theory. Moreover, since θ has negative mass dimensions, the action will also certainly be power-counting nonrenormalizable, whereas the full theory might be renormalizable [80, 34, 36, 108, 124].

Even more serious concerns emerge from the realization that the expansion one is performing is (of course) not truly based on a power series in the dimensionful quantity θ: it is rather an expansion in dimensionless quantities of the type $p\theta p$. Therefore already at tree level the truncated θ-expanded theory can only give a good approximation of the full theory at scales p such that $p\theta p \lesssim 1$, *i.e.* $p \lesssim 1/\sqrt{\theta}$.

But actually even in that range of momenta the expansion cannot be used reliably. Its reliability is spoiled by quantum corrections. The quantum corrections involve the Moyal $\star$-product inserted in loop diagrams, and the truncation will reliably describe these loop corrections only for loop momenta such that $p \lesssim 1/(\theta\Lambda)$. In fact, in loop integrals involving factors of the type $p\theta k$, with p playing the role of external momentum and k playing the role of integration/loop momentum, one would like a reliable truncation that is valid over the whole loop-integration range, which extends at least up to a cutoff Λ. In order to have $p\theta k \lesssim 1$ even for k as large as Λ it is necessary to assume that indeed $p \lesssim 1/(\theta\Lambda)$. This can also be inferred straightforwardly in the illustrative example of the "$\lambda\Phi^4$" scalar-boson field theory: there one finds that the full theory predicts nonplanar terms giving a leading contribution of the form

$$\Sigma^1_{NP}(p) \simeq \frac{g^2}{\widetilde{p}^2 + 1/\Lambda^2} = \Lambda^2 \frac{g^2}{\Lambda^2\widetilde{p}^2 + 1}. \tag{4.19}$$

whereas the truncated θ-expansion of the $\star$-product would replace this prediction with

$$\Sigma^1_{NP}(p) \simeq g^2\Lambda^2 \left\{1 - \Lambda^2\widetilde{p}^2 + O(\theta^4)\right\}. \tag{4.20}$$

Clearly the two expressions are equivalent only if $\Lambda^2\widetilde{p}^2 \lesssim 1$, which indeed corresponds to $p \lesssim 1/(\theta\Lambda)$.

Therefore, when one includes quantum/loop effects, the truncated θ-expansion could be a good approximation of the full theory only in the range of momenta $p \lesssim 1/(\theta\Lambda)$. But as we have discussed in the preceding section this is just the range of momenta in which the theory is maximally sensitive to ultraviolet physics, which we must assume to be unknown. In other words the truncated θ-expansion reliably approximates the full theory only in a regime where the full theory is itself void of predictive power,

because of its sensitivity to unknown physics that might be present in the ultraviolet. It therefore appears that these truncated θ-expansions cannot be used for a meaningful comparison between data and theories in canonical noncommutative spacetime. In other contexts expansions in powers of p versus some characteristic momentum scale have been proven to give a reliable low-energy effective-theory description of the full theory one intends to study, but in this case of field theories in canonical noncommutative spacetime the IR/UV mixing provides a powerful obstruction for any attempt to obtain a meaningful low-energy effective-theory description.

Chapter 5

CJT formalism for phase transition on CNC spacetime

* We have discussed how the IR/UV mixing, which significantly affects canonical noncommutative theories, causes strong IR problems. We have also emphasized that one manifestation of these IR problems (after the removal of the cut-off) is through zero-momentum poles in certain Green functions. IR problems of different origin but similar form, are known to plague also Thermal-Quantum-Field theories and have been successfully treated using a nonperturbative technique developed by Cornwall-Jackiw-Tomboulis (CJT). We apply the CJT formalism to the scalar $\lambda\varphi^4$ theory focusing in the so-called "bubble approximation". Assuming translational invariance of the vacuum we construct the gap equation and the CJT effective potential. We discuss the renormalizability of the CJT effective potential both in ordered and in the disordered phase for general values of the non-commutativity parameter θ. We comment in particular on the commutativity limit ($\theta \to 0$) and on the strong non-commutativity limit ($\theta \to \infty$). We observe that while in the disordered phase the hypothesis of translational invariance leads to a renormalizable effective potential, in the translational-invariant ordered phase, differently from the commutative case, the effective potential and the gap equation do not renormalize. We argue that our result, essentially based on a selective all-order resummation, appears to confirm the other (perturbative, one-loop) results, we reported in Chapter 3, that indicate the incompatibility of a translational-invariant ordered phase with the infrared structure of the canonical-noncommutative theories.

5.1 CJT formalism

In this section we briefly review the CJT formalism for the scalar theory in the commutative case [42, 43, 44]. The starting point is the definition of the partition function in the form

$$Z[J,K] = \exp W[J,K] = \int D\phi \exp\left\{S(\phi) + \int dx^4 J(x)\phi(x) + \int dy^4 dx^4 \phi(x)K(x,y)\phi(y)\right\},$$

[0]* In this Chapter we discuss in detail the analysis reported more briefly in Ref. [48].

in which two sources $J(x)$ and $K(x,y)$ are present.

One defines also $\varphi(x)$ and $G(x,y)$ by the relations $\frac{\delta W[J(x),K(x,y)]}{\delta J(x)} = \varphi(x)$ and $\frac{\delta W[J(x),K(x,y)]}{\delta K(x,y)} = \frac{1}{2}\{\varphi(x)\varphi(y) + G(x,y)\}$. Then one considers the double Legendre transformation of $W[J,K]$

$$\Gamma[\varphi(x), G(x,y)] = W[J(x), K(x,y)] - \int dx^4 \varphi(x) J(x) - \frac{1}{2}\int dx^4 dy^4 \varphi(x) K(x,y)\varphi(y) + \\ - \frac{1}{2}\int dx^4 dy^4 G(x,y) K(x,y) \tag{5.1}$$

which satisfies the relation

$$\frac{\delta\Gamma[\varphi,G]}{\delta\varphi(x)} = -J(x) + \int dy^4 K(x,y)\varphi(y),$$
$$\frac{\delta\Gamma[\varphi,G]}{\delta G(x,y)} = -\frac{1}{2}K(x,y).$$

The physical point corresponds to vanishing sources $K(x,y) = 0$, $J(x) = 0$, so that $\varphi(x)$ and $G(x,y)$ are solutions of the stationarity equations:

$$\frac{\delta\Gamma[\varphi,G]}{\delta\varphi(x)} = 0 \tag{5.2}$$

$$\frac{\delta\Gamma[\varphi,G]}{\delta G(x,y)} = 0 \tag{5.3}$$

It can be shown [42] that $\Gamma[\varphi, G]$ so defined is the generating functional for the two-particle irreducible(2PI) Green's functions, with propagator given by $G(x,y)$ and vertices given by $S_{int}(\varphi;\phi)$, where $S_{int}(\varphi;\phi)$ is obtained from $S(\varphi)$ by retaining only cubic and higher φ -terms in the expression of $S(\varphi+\phi)$.

One can expand (5.1) to obtain

$$\Gamma(\varphi, G) = S_{cl}(\varphi) - \frac{1}{2}TrLnD_0^{-1}G + \frac{1}{2}Tr\left\{D^{-1}G - 1\right\} + \Gamma^2(\varphi, G) \tag{5.4}$$

where

$$D^{-1}(x,y) = \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)},$$
$$D_0^{-1}(x,y) = D^{-1}|_{S_{free}},$$

and $\Gamma^2(\varphi, G)$ is the sum of vacuum diagrams(Fig.5.1) with vertices given by $S_{int}(\varphi;\phi)$ and propagators given by $G(x,y)$.

Example of two particle reducible graphs which do not contribute to $\Gamma^2(\varphi, G)$ are in Fig.5.2

Using (5.4) the gap equation (5.3) may be rewritten in the form

$$G^{-1}(x,y) = D^{-1}(x,y) + 2\frac{\delta\Gamma^2(\varphi,G)}{\delta G(x,y)}. \tag{5.5}$$

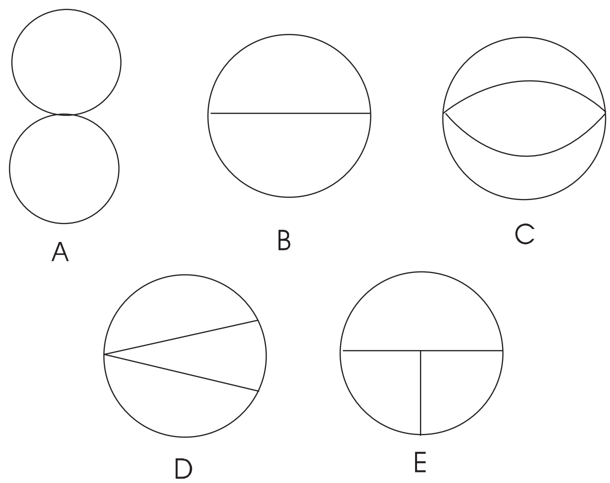

Figure 5.1: Two-particle irreducible graphs contributing to $\Gamma^2(\varphi, G)$ up to the three loop level.

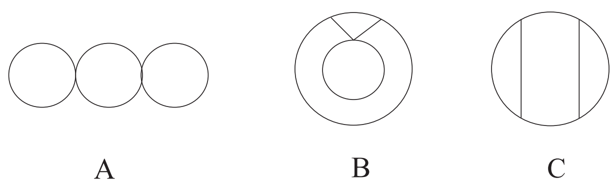

Figure 5.2: Examples of two-particle reducible graphs.

One can also recover the usual 1PI-effective action $\Gamma^{1PI}(\varphi)$ simply evaluating $\Gamma[\varphi, G]$ for vanishing $K(x, y)$:

$$\Gamma_{1PI}(\varphi) = \Gamma_{2PI}[\varphi, G_0], \tag{5.6}$$

where G_0 is solution of the gap equation

$$\frac{\delta\Gamma_{2PI}[\varphi, G]}{\delta G(x, y)} = 0. \tag{5.7}$$

This 2PI formalism which at a first sight might appear more involved than the standard 1PI formalism turns out to be very useful in certain calculations. This is the case for example of the so-called "bubble resummation", which means taking into account for all the diagrams generate by the vacuum to vacuum diagrams of the type of Fig.(5.3). In the case of the standard 1PI formalism the bubble resummation requires the evaluation of an infinite number of diagrams. In the 2PI-CJT formalism one obtains the whole "bubble resummation"([43]) simply considering the "eight"-diagram (A in Fig.5.1) contribution to $\Gamma^2(\varphi, G)$, and the corresponding gap equation. This "bubble resummation" turns out to be useful in theories in which the insertion of a tadpole is not effectively suppressed by the coupling constant as in the thermal field theories or canonical-noncommutative theories[1].

We will use this approximation in the following sections.

[1]In Thermal Field Theories the insertion of a tadpole which for example distingushes diagrams Fig.5.1 A and

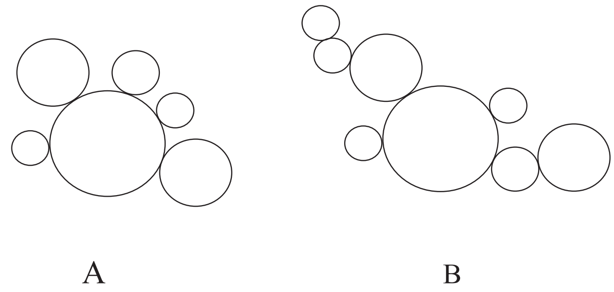

Figure 5.3: Vacuum to vacuum bubble diagrams: daisy (a) and super-daisy (b).

5.2 CJT formalism in canonical-noncommutative spacetime

We recall that once the Moyal $\star$-product (3.16) is introduced the scalar $\lambda\varphi^4$-theory in canonical noncommutative spacetime takes the form of a commutative theory with a deformed interaction given by substituting the products of fields with the $\star$-products. This implies that the CJT formalism should be applicable; in fact no specific assumptions are made in the CJT procedure about the form of the interaction. In particular Eq.(5.4) is still valid in our noncommutative context.

It is easy to see that $\lambda\varphi^4$-theory in canonical noncommutative spacetime

$$\begin{aligned} D^{-1}(x,y) &= \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)} = \\ &= -\left[\Box + m^2\right]_x \delta^4(x-y) + \\ &- \frac{\lambda}{3!}\left\{\delta^4(x-y)\star\varphi\star\varphi + \varphi\star\delta^4(x-y)\star\varphi + \varphi\star\varphi\star\delta^4(x-y)\right\}, \qquad (5.8) \\ D_0^{-1}(x,y) &= D^{-1}|_{S_{free}} = -\left[\Box + m^2\right]_x \delta^4(x-y) \qquad (5.9) \end{aligned}$$

As expected $D_0^{-1}(x,y)$ is not modified by noncommutativity since the integrals of terms quadratic in the fields are not modified by the $\star$-product (3.16), while $D^{-1}(x,y)$ acquires the θ-dependence. One can easily calculate the $\star$-products which appear in (5.8) obtaining, in the momentum space,

$$\delta^4(x-z)\star\varphi(x)\star\varphi(x) = \int dp_1^4 dp_2^4 dp_3^4 \exp F_1(\underline{p}) \exp[ix(p_1+p_2+p_3)] \exp[-ip_3 z]\widetilde{\varphi}(p_1)\widetilde{\varphi}(p_2)$$

$$\varphi(x)\star\delta^4(x-z)\star\varphi(x) = \int dp_1^4 dp_2^4 dp_3^4 \exp F_2(\underline{p}) \exp[ix(p_1+p_2+p_3)] \exp[-ip_3 z]\widetilde{\varphi}(p_1)\widetilde{\varphi}(p_2)$$

$$\varphi(x)\star\varphi(x)\star\delta^4(x-z) = \int dp_1^4 dp_2^4 dp_3^4 \exp F_3(\underline{p}) \exp[ix(p_1+p_2+p_3)] \exp[-ip_3 z]\widetilde{\varphi}(p_1)\widetilde{\varphi}(p_2)$$

Fig.5.2 A costs a factor $\lambda\frac{T^2}{m^2}$, which for large temperatures can be even large than 1. In noncommutative Thermal Field Theories the same insertion roughly comes with a factor $\lambda\frac{1}{p^2\theta^2 m^2}$ which, depending on the momentum entering the inserted tadpole, can be large.

where

$$F_1(\underline{p}) = -\frac{i}{2}\theta_{\mu\nu}\left\{p_1^\mu p_2^\nu + p_3^\mu p_1^\nu + p_3^\mu p_2^\nu\right\}$$
$$F_2(\underline{p}) = -\frac{i}{2}\theta_{\mu\nu}\left\{p_1^\mu p_3^\nu + p_1^\mu p_2^\nu + p_3^\mu p_2^\nu\right\}$$
$$F_3(\underline{p}) = -\frac{i}{2}\theta\left\{p_1^\mu p_2^\nu + p_1^\mu p_3^\nu + p_2^\mu p_3^\nu\right\}.$$

Now we have to calculate the S_{int} action of the 2PI formalism. One can proceed in two steps. The first step is the translation of the action $S(\phi) \to S(\phi+\varphi)$

$$S(\phi+\varphi) = \left\{\int dx^4 \frac{1}{2}m^2(\phi+\varphi)^2 + \frac{1}{2}\partial_\mu(\phi+\varphi)\partial^\mu(\phi+\varphi) + \right.$$
$$\left. + \frac{\lambda}{4!}(\phi+\varphi)\star(\phi+\varphi)\star(\phi+\varphi)\star(\phi+\varphi)\right\} \tag{5.10}$$

The second step is the one of retaining from (5.10) only cubic, and higher, terms in ϕ. So doing one obtains that the interaction vertices are given by the action

$$S_{int}(\phi;\varphi) = \frac{\lambda}{4!}\int dx^4 \phi\star\phi\star\phi\star\phi + \frac{\lambda}{6}\int dx^4 \phi\star\phi\star\phi\star\varphi$$

where the cyclitity of the $\star$-product (3.16) under integration has been used.

To proceed further we now need to adopt ansatz for the form of $G(x,y)$. Here we want to consider only transitionally invariant configurations so that the more general ansatz we can consider takes the form

$$G(x,y) = G(x-y) = \int d\alpha^4 \frac{\exp i\alpha(x-y)}{\alpha^2 + M^2(\alpha)} \tag{5.11}$$

where $M^2(\alpha)$ is to be determined. Once the ansatz (5.11) has been done we can start calculating the various terms in the left-hand-side of (5.4). The first term is trivial. The second term is

$$Tr\ln(D_0^{-1}G) = \int dx^4 \int dp^4 \ln\left\{\frac{p^2+m^2}{p^2+M^2(p)}\right\}.$$

The third term is

$$Tr\left\{D^{-1}G - 1\right\} = \int dx^4 \left\{\int dk^4 \frac{m^2 - M^2(k)}{k^2+M^2(k)} + \right.$$
$$\left. + \frac{\lambda}{3!}\int dp_1^4 dp_2^4 dp_3^4 F(\underline{p}) \exp ix(p_1+p_2)\widetilde{\varphi}(p_1)\widetilde{\varphi}(p_2)\frac{1}{p_3^2 + M^2(p_3)}\right\},$$

where $F(\underline{p}) = \sum_{i=1}^{3} \exp F_i(\underline{p})$.

Therefore we have that (5.4) with the ansatz (5.11) takes the form

$$\Gamma(\varphi, G) = \frac{1}{2}\int dx^4 \left\{(\partial_\mu\varphi)^2 + m^2\varphi^2\right\} + \frac{\lambda}{4!}\int dx^4 \varphi\star\varphi\star\varphi\star\varphi + \tag{5.12}$$
$$+ \frac{1}{2}\int dx^4 \int dp^4 \ln\left\{\frac{p^2+M^2(k)}{p^2+m^2}\right\} +$$
$$+ \frac{1}{2}\int dx^4 \int dk^4 \frac{m^2 - M^2(k) + \frac{\lambda}{3!}\int dp_1^4 dp_2^4 F(\underline{p},k)\exp ix(p_1+p_2)\widetilde{\varphi}(p_1)\widetilde{\varphi}(p_2)}{k^2+M^2(k)} +$$
$$+ \Gamma^2(\varphi, G)$$

Now we must evaluate $\Gamma^2(\varphi, G)$ with vertices given by $S_{int}(\phi; \varphi)$ and propagator given by $G(x, y)$.

As in the commutative case the CJT effective action can be used to obtain the 1PI effective action. In particular the one-loop 1PI effective action is obtained setting $\Gamma^2 = 0$. In this case the gap equation (5.5) reduces to

$$G^{-1}(x, y) = D^{-1}(x, y). \tag{5.13}$$

Using this expression one can easily compute the one-loop 1PI effective action in the 2PI CJT formalism. One has only to calculate

$$\left[D^{-1} D_0\right](x, y) = \delta^4(x - y) + K(x, y)$$

where

$$K(x, y) = \frac{\lambda}{3!} \int dz \left[{}^4\delta^4(x - z) \star \varphi \star \varphi + \varphi \star \delta^4(x - z) \star \varphi + \varphi \star \varphi \star \delta^4(x - z)\right] D(z - y). \tag{5.14}$$

Then from the usual expansion

$$\begin{aligned}
\Gamma(\varphi) &= S_{cl}(\varphi) + \frac{1}{2} Tr \ln\left[\delta^4(x - y) + K(x, y)\right] + O(\hbar^2) = \\
&= S_{cl}(\varphi) + \frac{1}{2} \sum_{i=1}^{\infty} \frac{(-1)^{n-1}}{n} Tr\left\{[K(x, y)]^n\right\} + O(\hbar^2) = \\
&= S_{cl}(\varphi) + \underbrace{\frac{1}{2} \int dx^4 K(x, x)}_{1-loop, 2legs} - \underbrace{...}_{1-loop, 4legs \text{ ecc}} + O(\hbar^2),
\end{aligned}$$

and from (5.14) follows that

$$\int dx^4 K(x, x) = \frac{\lambda}{3!} \int dp_1^4 dp_3^4 \left\{2 + \exp\left[-i\theta_{\mu\nu} p_1^\mu p_3^\nu\right]\right\} \widetilde{\varphi}(p_1) \widetilde{\varphi}(-p_1) \frac{1}{p_3^2 + M^2}, \tag{5.15}$$

so that

$$\Gamma(\varphi) = S_{cl}(\varphi) + \frac{\lambda}{3!} \int dp_1^4 dp_3^4 \widetilde{\varphi}(p_1) \widetilde{\varphi}(-p_1) \left\{2 + \exp\left[-i\theta_{\mu\nu} p_1^\mu p_3^\nu\right]\right\} \frac{1}{p_3^2 + M^2} + ...$$

That is the 1PI-effective action up to the one-loop corrected quadratic terms.

5.3 The effective potential

While neglecting Γ^2 completely simply gives us back the one-loop 1PI effective action, a more interesting results is obtained by approximating $\Gamma^2(\varphi, G)$ including only the "eight" diagram (A in Fig.5.1). In this approximation one has that

$$\Gamma^2(\varphi, G) = \frac{1}{4!} \lambda \delta^4(0) \int d\alpha^4 d\alpha'^4 \frac{1}{\alpha^2 + M^2(\alpha)} \frac{1}{\alpha'^2 + M^2(\alpha')} \left\{1 + 2\cos^2\left(\frac{\alpha\theta\alpha'}{2}\right)\right\}. \tag{5.16}$$

We observe that differently from the commutative case, where the momenta circulating in each of the two loops do not mix, in the noncommutative case this mixing occurs.

From the effective action(5.12) and (5.16), assuming that, as consequence of the translational invariance of the vacuum, $\varphi(x) = \varphi$ one can extract the potential $V(\varphi, G) = \Gamma^{2PI}(\varphi, G)/\int dx^4$

$$\begin{aligned}
V(\varphi) = & \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \\
& + \frac{1}{2}\int dp^4 \ln\left\{\frac{p^2 + M^2(p)}{p^2 + m^2}\right\} + \\
& + \frac{1}{2}\int dk^4 \frac{m^2 - M^2(k) + \frac{\lambda}{2}\varphi^2}{k^2 + M^2(k)} + \\
& + \frac{1}{4!}\lambda \int d\alpha^4 d\alpha'^4 \frac{1}{\alpha^2 + M^2(\alpha)} \frac{1}{\alpha'^2 + M^2(\alpha')} \left\{2 + \cos(\alpha\theta\alpha')\right\}.
\end{aligned} \tag{5.17}$$

The stationarity conditions (5.2) and (5.3) in this case read

$$0 = \frac{\partial V(\varphi, G)}{\partial \varphi} = \varphi\left[m^2 + \frac{\lambda}{3!}\varphi^2 + \frac{\lambda}{2}\int dk^4 \frac{1}{k^2 + M^2(k)}\right], \tag{5.18}$$

$$0 = \frac{\partial V(\varphi, G)}{\partial M^2} = M^2(\alpha) - m^2 - \frac{\lambda}{3!}\varphi^2 - \frac{\lambda}{6}\int db^4 \frac{1}{b^2 + M^2(b)} \left\{2 + \cos(b\theta\alpha)\right\}. \tag{5.19}$$

The first equation has the solutions $\varphi = 0$ and $m^2 = -\frac{\lambda}{3!}\varphi^2 - \frac{\lambda}{2}\int dk^4 \frac{1}{k^2+M^2(k)}$ which correspond respectively to the symmetric phase and to the broken-symmetric phase. Substituting the gap equation (5.19) in the expression of the potential (5.17) we obtain

$$\begin{aligned}
V(\varphi, G) = & \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\int dp^4 \ln\left\{\frac{p^2 + M^2(p)}{p^2 + m^2}\right\} + \\
& - \frac{\lambda}{24}\int dk^4 \frac{1}{k^2 + M^2(k)} \int db^4 \frac{1}{b^2 + M^2(b)} \left\{2 + \cos(b\theta k)\right\}
\end{aligned} \tag{5.20}$$

The term in the second row of the above expression is generated by the "bubble summation". The terms appearing in the first row are already present at the tree level and at one loop but what is different here is that they now must be evaluated for $M^2(p)$ solution of the gap equation (5.19), which under the hypothesis of translational invariance, takes the form

$$M^2(\alpha) = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{6}\int db^4 \frac{1}{b^2 + M^2(b)} \left\{2 + \cos(b\theta\alpha)\right\}. \tag{5.21}$$

Both (5.20) and (5.21) are ultraviolet divergent and they both are considered to be regularized with a cutoff Λ on the loop-momenta. In the next section we will deal with the problem of their renormalization.

5.3.1 Commutativity limit

In this section we want discuss the commutativity limit ($\theta \to 0$) and the strong noncommutativity limit ($\theta \to \infty$). In the commutativity limit eqs.(5.19) and (5.20) become respectively

$$V(\varphi, G) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\int dp^4 \ln\left\{\frac{p^2 + M^2(p)}{p^2 + m^2}\right\} - \frac{\lambda}{8}\int dk^4 \frac{1}{k^2 + M^2(k)}\int db^4 \frac{1}{b^2 + M^2(b)} \tag{5.22}$$

and

$$M^2(\alpha) = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{2}\int db^4 \frac{1}{b^2 + M^2(b)}, \tag{5.23}$$

which is the well known result of [44]. We recall the procedure that one can use to renormalize (5.22) and (5.23) since we will use it widely in the rest of the chapter. The gap equation (5.23) can be renormalized in the following way [44]

$$\frac{M^2}{\lambda} = \frac{m^2}{\lambda} + \frac{1}{2}\varphi^2 + \frac{1}{2}\int db^4 \frac{1}{b^2 + M^2},$$

$$\int db^4 \frac{1}{b^2 + M^2} = I_1 - I_2 M^2 + G_R(M),$$

$$M^2\left(\frac{1}{\lambda} + \frac{1}{2}\int db^4 \frac{1}{b^4}\right) = \frac{m^2}{\lambda} + \int db^4 \frac{1}{b^2} + \frac{1}{2}\varphi^2 + \frac{1}{3}G_R(M),$$

where $G_R(M)$ is the finite part of $G(M)$, $I_1 = \int db^4 \frac{1}{b^2}$,and $I_2 = \int db^4 \frac{1}{b^4}$ Introducing the renormalized parameters

$$\frac{1}{\lambda_R} = \lim_{\Lambda\to\infty}\left[\frac{1}{\lambda} + \frac{1}{2}I_1\right], \tag{5.24}$$

$$\frac{m_R^2}{\lambda_R} = \lim_{\Lambda\to\infty}\left[\frac{m^2}{\lambda} + \frac{1}{2}I_2\right], \tag{5.25}$$

one gets the renormalized gap equation

$$M^2 = m_R^2 + \frac{\lambda_R}{2}\varphi^2 + \frac{\lambda_R}{3}G_R(M).$$

One would like obtain a renormalized effective potential written in terms of M^2. Using (5.23) one can write (5.22) as the sum of the three contributions

$$V^0 = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4, \tag{5.26}$$

$$V^I = \frac{1}{2}\int dk^4 \ln(k^2) + \frac{1}{2}I_1 M^2 - \frac{1}{4}I_2 M^4 + T \tag{5.27}$$

$$V^{II} = -\frac{1}{2}GM^2 + \frac{1}{2\lambda}M^4 - \frac{1}{2\lambda}m^4 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{8}\varphi^4, \tag{5.28}$$

and one finds that up to a φ-independent term which of course can we ignored in effective potential analyses

$$V = V^{II} + V^I + V^0 = -\frac{\lambda}{12}\varphi^4 + \frac{1}{2}\frac{M^4}{\lambda_R} - \frac{1}{2}M^2 G_R + \frac{1}{2}\int dk^4 \ln(k^2) + T \tag{5.29}$$

where T is the finite part of the expansion of V^I (which of course does not play an important role in renormalization).

We observe that (5.29) is finite if written in terms of the renormalized parameters defined by (5.24) and (5.25). In particular it is worth noticing the exact cancellation of the divergent terms in $m^2\varphi^2$ which appear with opposite signs in the tree-level contribution V^0 and in the loop correction V^{II}.

5.3.2 Strong noncommutativity limit

Now we analyze the limit of strong noncommutativity ($\theta \to \infty$). In this limit the strong oscillations in the phases, which are present in the integrands of (5.20) and (5.21), induce the vanishing of the corresponding integrals. The effective potential and the gap equation in this case become respectively

$$V(\varphi, G) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\int dp^4 \ln\left\{\frac{p^2+M^2}{p^2+m^2}\right\} - \frac{\lambda}{12}\int dk^4 \frac{1}{k^2+M^2}\int db^4 \frac{1}{b^2+M^2} \quad (5.30)$$

and

$$M^2 = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{3}\int db^4 \frac{1}{b^2+M^2}. \quad (5.31)$$

We observe that these expressions are formally similar to the ones of the commutative case (5.22), (5.23). The only differences are in the terms in front of the eight-diagram contribution: the nonplanar the eight-diagram contribution in fact becomes negligible in the $\theta \to \infty$ limit and only contributions of the planar diagrams survive. It is also important to notice that the dressed mass $M^2(k)$ become in this limit momentum independent and one can follow exactly the same procedure as in the commutative case. One obtains for the gap equation

$$M^2 = m_R^2 + \frac{\lambda_R}{2}\varphi^2 + \frac{\lambda_R}{3}G_R(M).$$

where now

$$\frac{1}{\lambda_R} = \lim_{\Lambda\to\infty}\left[\frac{1}{\lambda} + \frac{1}{3}\int db^4 \frac{1}{b^4}\right], \quad (5.32)$$

$$\frac{m_R^2}{\lambda_R} = \lim_{\Lambda\to\infty}\left[\frac{m^2}{\lambda} + \frac{1}{3}\int db^4 \frac{1}{b^2}\right]. \quad (5.33)$$

The effective potential is obtained again as the sum of the terms

$$V^0 = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 \quad (5.34)$$

$$V^I = \frac{1}{2}\int dk^4 \ln(k^2) + \frac{1}{2}I_1 M^2 - \frac{1}{4}I_2 M^4 + T \quad (5.35)$$

$$V^{II} = -\frac{3}{4\lambda}m^4 - \frac{3\lambda}{16}\varphi^4 + \frac{3}{4\lambda}M^4 - \frac{1}{2}M^2 G - \frac{3}{4}m^2\varphi^2. \quad (5.36)$$

which gives

$$V = V^{II} + V^{I} + V^{0} = -\frac{1}{4}m^2\varphi^2 - \frac{7\lambda}{48}\varphi^4 + \frac{3}{4}\frac{M^4}{\lambda_R} + \frac{1}{3}\int dk^4 \frac{1}{k^4} + \frac{1}{2}\int dk^4 \ln[k^2] - \frac{3}{4\lambda}m^4$$

We observe that in the case of strong noncommutativity the cancellation between the $m^2\varphi^2$ terms in (5.34) and (5.36) does not occur and the resulting potential does not renormalize.

5.3.3 Effective potential in the general case

We have so far discussed the limits of commutativity ($\theta \to 0$) and strong noncommutativity ($\theta \to \infty$). We have seen that in both of these cases the unknown function $M^2(k)$, that appears in the denominator of $G(M)$, is momentum independent, although it satisfies different gap equations in the two different cases. Now we want to address the problem of calculating the effective potential for general values of the noncommutativity parameter θ. We start by defining $M^2(\alpha) = M^2 + \Pi(\alpha)$ so that the gap equation (5.21) can be rewritten as

$$\Pi(\alpha) = -M^2 + m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{3}\int db^4 \frac{1}{b^2 + M^2 + \Pi(b)} + \frac{\lambda}{6}\int db^4 \frac{\cos(b\theta\alpha)}{b^2 + M^2 + \Pi(b)}.$$

The last equation must hold for every value of α and θ. Thus we must have that separately

$$-M^2 + m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{3}\int db^4 \frac{1}{b^2 + M^2 + \Pi(b)} = \mathrm{C}, \tag{5.37}$$

and

$$\Pi(\alpha) - \frac{\lambda}{6}\int db^4 \frac{\cos(b\theta\alpha)}{b^2 + M^2 + \Pi(b)} = \mathrm{C}, \tag{5.38}$$

where C is α and θ independent.

We observe that we can always choose C=0, modulo the redefinitions $\Pi(\alpha) \to \Pi(\alpha)+\mathrm{C}$, $M^2 \to M^2-\mathrm{C}$. In this way we obtain

$$M^2 = m^2 + \frac{\lambda}{2}\varphi^2 + \frac{\lambda}{3}\int db^4 \frac{1}{b^2 + M^2 + \Pi(b)}, \tag{5.39}$$

which determines M^2, and

$$\Pi(\alpha) = \frac{\lambda}{6}\int db^4 \frac{\cos(b\theta\alpha)}{b^2 + M^2 + \Pi(b)} \tag{5.40}$$

which determines $\Pi(\alpha)$.

We see from (5.40) that $\Pi(\alpha) \to \infty$ for $\alpha \to 0$ and for $\theta \to 0$, and that $\Pi(\alpha) \to 0$ for $\alpha \to \infty$ and for $\theta \to \infty$. Equation (5.39) can be renormalized by the following procedure, similar to the one we have discussed previously

$$\frac{M^2}{\lambda} = \frac{m^2}{\lambda} + \frac{1}{2}\varphi^2 + \frac{1}{3}\int db^4 \frac{1}{b^2 + M^2 + \Pi(b)},$$

$$\int db^4 \frac{1}{b^2 + M^2 + \Pi(b)} = \int db^4 \frac{1}{b^2 + \Pi(b)} - M^2 \int db^4 \frac{1}{[b^2 + \Pi(b)]^2} + G_R(M),$$

$$M^2\left(\frac{1}{\lambda}+\frac{1}{3}\int db^4\frac{1}{[b^2+\Pi(b)]^2}\right)=\frac{m^2}{\lambda}+\int db^4\frac{1}{b^2+\Pi(b)}+\frac{1}{2}\varphi^2+\frac{1}{3}G_R(M),$$

where $G_R(M)$ is the finite part of the divergent expression in (5.39). We can now introduce the renormalized parameters in the form

$$\frac{1}{\lambda_R}=\lim_{\Lambda\to\infty}\frac{1}{\lambda}+\frac{1}{3}\int db^4\frac{1}{[b^2+\Pi(b)]^2}, \tag{5.41}$$

$$\frac{m_R^2}{\lambda_R}=\lim_{\Lambda\to\infty}\frac{m^2}{\lambda}+\int db^4\frac{1}{b^2+\Pi(b)}, \tag{5.42}$$

and we get the renormalized gap equation

$$M^2=m_R^2+\frac{\lambda_R}{2}\varphi^2+\frac{\lambda_R}{3}G_R(M)$$

We come now to the important issue of the renormalization of the effective potential. We have seen that the way in which the gap equation renormalizes fixes uniquely the renormalization of the bare mass and the renormalization of the coupling. We must check if the same renormalization conditions provide us with a finite effective potential. We can use (5.39) and (5.40) in the expression (5.20) to obtain for the effective potential

$$\begin{aligned}
V&=V^0+V^I+V^{II}=\\
&=\frac{M^4}{4\lambda_R}+M^2\left\{-\frac{1}{2}\int dk^4R(k)+\frac{1}{4}\int dk^4\frac{\Pi(k)}{[k^2+\Pi(k)]^2}\right\}+\\
&-\frac{1}{4}\int dk^4\frac{\Pi(k)}{k^2+\Pi(k)}-\frac{1}{4}\int dk^4\Pi(k)R(k)+\frac{1}{2}\int dk^4\ln[1+\frac{\Pi(k)}{k^2}]+T+\\
&-\frac{3m^4}{4\lambda}-\frac{1}{4}m^2\varphi^2-\frac{7}{48}\lambda\varphi^4+\frac{1}{2}\int dk^4\ln[k^2],
\end{aligned} \tag{5.43}$$

where we used

$$V^0=\frac{1}{2}m^2\varphi^2+\frac{\lambda}{4!}\varphi^4, \tag{5.44}$$

$$V^I=\frac{1}{2}\int dk^4\ln(k^2)+\frac{1}{2}I_1M^2-\frac{1}{4}I_2M^4+T, \tag{5.45}$$

$$\begin{aligned}
V^{II}&=M^4\left\{\frac{1}{2}\int db^4\frac{1}{[k^2+\Pi(k)]^2}+\frac{3}{4}\frac{1}{\lambda}\right\}+\\
&+M^2\left\{-\frac{1}{2}\int dk^4R(k)-\frac{1}{2}\int dk^4\frac{1}{k^2+\Pi(k)}+\frac{1}{4}\int dk^4\frac{\Pi(k)}{[k^2+\Pi(k)]^2}\right\}+\\
&-\frac{1}{4}\int dk^4\frac{\Pi(k)}{k^2+\Pi(k)}-\frac{1}{4}\int dk^4\Pi(k)R(k)-\frac{3m^4}{4\lambda}-\frac{3}{4}m^2\varphi^2-\frac{3}{16}\lambda\varphi^4,
\end{aligned} \tag{5.46}$$

and we defined

$$\begin{aligned}
T&=\frac{1}{2}\int dk^4\ln[k^2+M^2+\Pi(k)]+\\
&-\left\{\frac{1}{2}\int dk^4\ln[k^2+\Pi(k)]+\tfrac{M^2}{2}\int dk^4\tfrac{1}{k^2+\Pi(k)}-\tfrac{M^4}{4}\int dk^4\tfrac{1}{[k^2+\Pi(k)]^2}\right\}
\end{aligned}$$

and

$$R(k) = \frac{1}{k^2+M^2+\Pi(k)} - \left\{ \frac{1}{k^2+\Pi(k)} - \frac{M^2}{[k^2+\Pi(k)]^2} \right\}. \tag{5.47}$$

We observe that in (5.43) all the field-dependent terms, with the exception of $-\frac{1}{4}m^2\varphi^2$, are finite thanks to the fact that $\Pi(k)$ vanishes exponentially in the limit $k \to \infty$. As in the case of strong noncommutativity limit, the presence of the divergent term $-\frac{1}{4}m^2\varphi^2$ is due to the fact that the corresponding contributions from V^0 and V^{II} do not cancel each other. The cancellation occurs only in the commutative limit.

5.4 Remarks on the structure of the CJT effective potential in canonical noncommutative spacetime

Whereas in commutative spacetime the CJT effective potential can be renormalized and gives a satisfactory description of the vacua of a given field theory, in our canonical-noncommutativity analysis the CJT effective potential (in the bubble-resummation approximation) was found not to be renormalizable. From a conservative standpoint we should then assume that in this type of theories the CJT effective potential cannot provide reliable nonperturbative insight on the phase structure. This negative conclusion is also supported by the realization that canonical noncommutativity affects strongly the structure of the UV divergences of a field theory, and this might be particularly significant for those techniques that effectively rely on resummations of contributions from all orders in the coupling constant. When we establish that a field theory is renormalizable, we actually verify that it is "perturbatively renormalizable": the divergences at any given order in coupling-expansion perturbation theory can be reabsorbed in redefinitions of the parameters of the Lagrangian density. The fact that the CJT technique gives rise to a renormalizable effective potential in the commutative-spacetime case is highly nontrivial, since we are not consistently summing all contributions up to a given order in the coupling constant (a calculation which would be "protected" by peturbative renormalizability), we are instead selectively summing a certain subset of the contributions at each order in the coupling constant. It is therefore plausible that the fact that our CJT effective potential cannot be renormalized is simply a sign of an inadequacy of this technique to the canonical-noncommutativity context. On the other hand it appears reasonable to explore an alternative, more optimistic, perspective, which is based on the observation that the only contribution to the CJT potential that ends up not being expressed in terms of renormalized quantities does not is the term $\frac{1}{4}m^2\varphi^2$. This term however vanishes in the disordered phase $\varphi = 0$. In a certain sense we have a renormalizable effective potential in the disordered phase, and our results of nonrenormalizability in the translationally-invariant ordered phase $\varphi = C$ could be interpreted as a manifestation of the fact that this phase is not admissable for these theories in canonical noncommutative spacetime. This hypothesis finds some support in the arguments presented in Ref. [115], which also

concluded that the only admissable phases for these theories are the disordered phase and a (non-translationally-invariant) stripe phase with $\tilde{\varphi}(p) = C\delta(p - p_c)$ (where $\tilde{\varphi}(p)$ is the Fourier transform of $\varphi(x)$) and p_c is a characteristic momentum scale of the stipe phase). This argument of inadmissability of the translationally-invariant ordered phase might be related with the delicate IR structure of these theories: $\varphi(x) = C$ means $\tilde{\varphi}(p) = C\delta(p)$, so the concept of a translationally-invariant ordered phase is closely connected with the zero-momentum structure of the theory of interest. To explore these issues it would be necessary to consider the CJT effective action, which explores the more general class of candidate vacua $\varphi(x)$, rather than stopping, as we did here, at the level of the CJT effective potential (which assumes from the beginning a translationally-invariant vacuum). With the CJT effective action one could investigate the renormalizability of the stripe phase (which is not translationally invariant, and therefore cannot be studied with the effective potential). Moreover, while the effective potential is the generating functional of Green functions at zero external momentum, could be particularly sensitive the effective action is the generating functional of generic Green functions and might be less sensitive to the troublesome IR sector of these theories. The analysis of the CJT effective action is postponed to a future study. Even in commutative-spacetime theories the evaluation of the CJT effective action turns out to be very complex, basically intractable analytically, and a troublesome calculation even numerically. It is likely that in the canonical-noncommutativity context the evaluation of the CJT effective action may prove even more troublesome, but from the indications that emerged from our analysis of the CJT effective potential it appears that such an analysis is well motivated, as it could provide insight for the understanding of some key physical predictions of these theories.

Chapter 6

Conclusions

It this thesis, we have explored the hypothesis that nonclassical effects of spacetime may manifest through the noncommutativity of spacetime at short distances. We focused on the two most popular examples of noncommutative spacetimes: canonical spacetimes, which have been at the center of an intense scientific debate over the last few years (mostly because of their relevance for the description of string theory in certain backgrounds) and κ-Minkowski spacetime, which, being the only fully-worked-out example of spacetime requiring a Planck-scale "deformation" of Poincaré symmetries, is also being investigated by a large number of research groups.

We focused on some issues that provide key physical characterizations of these spacetimes. In the light of the fact that plans for experimental searches of a possible dependence of the group velocity on the Planck scale are already at an advanced stage [25, 73, 72], we analyzed wave propagation both in canonical and in κ-Minkowski spacetime. The idea that the Planck-scale (quantum) structure of spacetime might affect the group-velocity/wavelength relation is plausible (and in some cases inevitable) in most quantum-gravity approaches, including phenomenological models of spacetime foam [17], loop quantum gravity (see, *e.g.*, Ref. [125]), superstring theory (see, *e.g.*, Ref. [35]), and noncommutative geometry. While a detailed careful description of wave propagation is beyond the reach of the present technical understanding of most quantum-gravity scenarios, we showed here that wave propagation in certain noncommutative spacetimes can be rigorously analyzed. We have shown that the features of the propagating waves strongly depend on the type of noncommutative spacetime one is considering. In the case of waves in canonical spacetime we found no observable departure from the classical picture of propagating wave. Instead, in the case of waves in κ-Minkowski spacetime, our analysis showed that the group velocity is affected by noncommutativity. We found that the formula $v = dE(p)/dp$, where $E(p)$ is fixed by the κ-Poincaré dispersion relation, still holds in κ-Minkowski spacetime (just like $v = dE(p)/dp$ holds in the Galilei/Minkowski classical spacetimes and in the canonical noncommutative spacetime) but it actually sets a new type of relation between group velocity

and momentum as a result of the fact that in κ-Minkowski the dispersion relation $E(p)$ is fixed by the κ-Poincaré mass Casimir, which differs from the familiar Poincaré mass Casimir at the level of Planck-scale-suppressed effects.

The validity of $v = dE(p)/dp$ in κ-Minkowski had been largely expected in the literature, even before our direct analysis, but such a direct analysis had become more urgent after the appearance of some recent articles [27, 28] which had argued in favor of alternatives to $v = dE(p)/dp$ for κ-Minkowski. We have shown that these recent claims were incorrect: the analysis reported in Ref. [28] was based on erroneous implementation of the κ-Minkowski differential calculus, while the analysis in Ref. [27] interpreted as momenta some quantities which cannot be properly described in terms of translation generators.

Of course, one is interested in going much beyond the description of wave propagation: a key objective for this research field is the construction and analysis of quantum field theories in these noncommutative spacetimes. As discussed in Chapter 3, the path toward the construction of a sensible quantum field theory in κ-Minkowski spacetime appears to be still confronted with a large number of delicate obstacles. Our analysis of wave propagation in κ-Minkowski is therefore the (very limited) "state of the art" in the analysis of the physical predictions of this noncommutative spacetime. Since a key ingredient of a quantum field theory is essentially an expansion of fields in plane waves, perhaps the results that emerged from our rigorous analysis of wave propagation in κ-Minkowski spacetime could prove useful for an improved formulation of QFT in this spacetime but at present we do not see an obvious way to approach this project.

For canonical noncommutative spacetimes there is instead a much studied approach to the construction of quantum field theories, based on the $\star$-product technique. In this framework it appears that several important issues are entangled with the peculiar failure of Wilson decoupling between infrared and ultraviolet degrees of freedom. In particular, as we showed in Chapter 4, the IR/UV mixing has wide implications for the strategies that should be adopted in order to falsify/verify these theories. Theories that (according to our conventional perspective) differ only in an experimentally unaccessible range of momenta may give rise to different predictions in the low-energy regime. In fact we found that predictions for the low-energy (i.e. experimentally accessible) physics depend strongly not only on the low-energy structure of the theory but also on its high-energy structure. Therefore the bounds on parameter space that one usually is able to set using low-energy data are here only "conditional". A comparison between low-energy data and the low-energy sector of theory can be reliably done only after having fully specified the high-energy sector.

While in Chapter 4 we explored the consequences of the IR/UV mixing using a standard perturbative Feynman-diagrammatic approach, in Chapter 5 we used the Cornwall-Jackiw-Tomboulis formalism, a nonperturbative technique (effectively resumming infinite series of 1PI

Feynman diagrams) which is usually very fruitful in contexts in which the infrared sector is problematic (as in Thermal Quantum Field Theories [42, 43, 44]). We studied the CJT effective potential in the canonical-noncommutative "$\lambda\varphi^4$" theory, adopting the so-called "bubble-resummation approximation", which includes all contributions from daisy and super-daisy diagrams. We found that the effective potential is in general nonrenormalizable, but the left-over UV divergences disappear in the disordered phase $\varphi = 0$. We argued that the nonrenormalizability of the effective potential might be another manifestation of the IR/UV mixing, and that the problems with the translationally-invariant ordered phase $\varphi =$C might also be due to the delicate IR structure. Probably the infrared structure of these theories prohibits the condensation of the zero-momentum modes which is necessary requirement for a transition to a transitionally-invariant vacuum. A characteristic prediction of these theories might therefore be the presence of energetically-favored transitionally-invariant phases, such as the stripe phases mentioned in Sec. 5.4.

Among the investigations that could find motivation in the results here reported we emphasize the study of the CJT effective action in theories in canonical noncommutative spacetime, which could bring key insight on the phase structure of these theories.

It would also be interesting to explore new strategies for setting limits on the canonical-noncommutativity parameters, since, as shown in Chapter 4, the standard strategy is unreliable. Perhaps the only possibility is the one of "building a case" in favor or against consistency with observations (whereas in commutative spacetime a single experiment can give conclusive unconditional indications). The case would be built by considering a variety of data, and observing that they are all consistent with the characteristic structure of theories in canonical noncommutative spacetime. Because of the nature of these characteristic features of canonical noncommutativity, it might be useful to rely on data that concern a wide range of energy scales. The astrophysical studies analyzed, for what concerns canonical noncommutativity, in Ref. [26] could play an important role in this programme.

Appendix A

Hopf algebras

In this appendix we report the basic definitions and the basic properties of the Hopf algebras that have been used in this thesis.

Definition 1 *An algebra $\mathcal{A}$ is a vector space (over the field $\mathcal{C}$) in which two linear applications m: $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\eta : \mathcal{C} \to \mathcal{A}$ are defined such that*

$$m \circ (id \otimes m) = m \circ (m \otimes id) \tag{A.1}$$

$$m \circ (\eta \otimes id) = m \circ (id \otimes \eta) = id \tag{A.2}$$

where $\circ$ indicates the composition of applications and id is the identity application.

The property (A.1) is the associativity of the product m. The property (A.2) is the neutrality of the identity with respect to the product m.

Definition 2 *A coalgebra $\mathcal{B}$ is a vector space (over the field $\mathcal{C}$) in which two linear maps $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ (the 'coproduct') and $\varepsilon : \mathcal{B} \to \mathcal{C}$ (the 'counit') are defined such that*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \tag{A.3}$$

$$(\varepsilon \otimes id) \circ \Delta = (id \otimes \varepsilon) \circ \Delta = id. \tag{A.4}$$

The property (A.3) is the (co-)associativity of the coproduct Δ. The property (A.4) is the (co-)neutrality of the identity with respect to the coproduct Δ.

It is worth noticing that a coalgebra ($\mathcal{B}$) can always be constructed from an algebra ($\mathcal{A}$) by duality (i.e. using a nondegenerate map $\langle , \rangle : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$) in the following way.

1. First, one renders $\mathcal{B}$ a vector space by the definitions

$$\langle a, b + c\rangle := \langle a, b\rangle + \langle a, c\rangle \qquad \forall a \in \mathcal{A}, \forall b, c \in \mathcal{B}, \tag{A.5}$$

$$\langle a, \lambda b\rangle := \lambda \langle a, b\rangle \qquad \forall a \in \mathcal{A}, \forall b \in \mathcal{B},\ \forall \lambda \in \mathcal{C}. \tag{A.6}$$

2. Secondly, one defines the coproduct $\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ and the counit $\varepsilon : \mathcal{B} \to \mathcal{C}$ by the relations

$$\langle a \otimes b, \Delta c \rangle := \langle ab, c \rangle \,, \tag{A.7}$$

$$\varepsilon(c) := \langle \mathbb{I}_{\mathcal{A}}, c \rangle \,, \tag{A.8}$$

where $\langle , \rangle$ extends to tensor products pairwise (i.e. $\langle a \otimes b, \Delta c \rangle = \left\langle a, c_{(1)} \right\rangle \left\langle b, c_{(2)} \right\rangle$).

It is easy to verify that Δ and ε defined by (A.7,A.8) satisfy (A.3,A.4), so that $\mathcal{B}$ is a coalgebra.

Definition 3 *$(\mathcal{H}, m, \eta, \Delta, \varepsilon)$ is a bialgebra if*

1. *$\mathcal{H}$ is an algebra with respect to m, η.*

2. *$\mathcal{H}$ is a coalgebra with respect to Δ, ε.*

3. *Δ and ε obey the following relations:*

$$\Delta(ab) = \Delta(a)\Delta(b) \qquad \forall a, b \in \mathcal{A}, \tag{A.9}$$

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \qquad \forall a, b \in \mathcal{A}, \tag{A.10}$$

where the tensor product $(h \otimes g)(h' \otimes g') := m(h, h') \otimes m(g, g')$ is defined $\forall h, h', g, g' \in \mathcal{H}$.

The property (A.9) means that the coproduct (Δ) furnishes a representation of the product (m) over $\mathcal{H} \otimes \mathcal{H}$. The property (A.10) means that the counit (ε) furnishes a representation of the product (m) over $\mathcal{C}$.

Definition 4 *An Hopf algebra is a bialgebra $(\mathcal{H}, m, \eta, \Delta, \varepsilon)$ with a map $S : \mathcal{H} \to \mathcal{H}$ (the 'antipode') such that*

$$m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta. \tag{A.11}$$

From (A.11) it follows that

$$S(ab) = S(a)S(b) \qquad \forall a, b \in \mathcal{H}, \tag{A.12}$$

and that

$$\Delta(S(a)) = S(a_{(2)}) \otimes S(a_{(1)}) \qquad \forall a \in \mathcal{H}. \tag{A.13}$$

The property (A.12) says that the antipode (S) furnishes a representation of the product (m) over $\mathcal{H}$. The property (A.13) says that the antipode (S) furnishes an (anti-)representation of the coproduct (Δ) over $\mathcal{H} \otimes \mathcal{H}$.

Definition 5 *An Hopf algebra is commutative if it is commutative as an algebra. It is 'cocommutative' if it is cocommutative as a coalgebra (i.e. if $\tau \circ \Delta = \Delta$, where τ is the 'flip operator' defined by $\tau(a_{(1)} \otimes a_{(2)}) := a_{(2)} \otimes a_{(1)}$).*

It is also easy to verify that

Corollary 6 *If $\mathcal{H}$ is a commutative or cocommutative Hopf algebra, then $S^2 = id$.*

We notice that the key difference between a "classical" group and a truly "quantum" group regards the coalgebra sector: while the former is cocommutative, the latter is noncocommutative.

Definition 7 *Two Hopf algebras $\mathcal{H}$ and $\mathcal{H}$' are 'dually paired' by a map $\langle , \rangle : \mathcal{H} \otimes \mathcal{H}' \rightarrow \mathcal{C}$ if*

$$\langle \phi\psi, h\rangle = \langle \phi \otimes \psi, \Delta h\rangle \,, \qquad \langle 1, h\rangle = \epsilon(h), \tag{A.14}$$

$$\langle \Delta\phi, h \otimes g\rangle = \langle \phi, hg\rangle \,, \qquad \langle \phi, 1\rangle = \epsilon(\phi), \tag{A.15}$$

$$\langle S\phi, h\rangle = \langle \phi, Sh\rangle \,, \tag{A.16}$$

for all $\phi, \psi \in \mathcal{H}$' and $h, g \in \mathcal{H}$. Here $\langle , \rangle$ extends pairwise to tensor products.

This definition implies that the product of $\mathcal{H}$ and the coproduct of $\mathcal{H}'$ are adjoint to each other under $\langle , \rangle$, and vice-versa. Likewise, the units and counits are mutually adjoint, and the antipodes are adjoint.

Definition 8 *A bialgebra or Hopf algebra $\mathcal{H}$ acts*[1] *on an algebra $\mathcal{A}$ if a linear map (the action) is defined $\triangleright : \mathcal{H} \otimes \mathcal{A} \rightarrow \mathcal{A}$ such that*

$$(ab) \triangleright c = a \triangleright (b \triangleright c), \qquad \forall a, b \in \mathcal{H}, \quad \forall c \in \mathcal{A}. \tag{A.17}$$

The notion of action is the generalization of the notion of linear transformation to the Hopf algebras.

Definition 9 *Given two Hopf algebras $\mathcal{H}$ and $\mathcal{A}$, an action $\triangleright$ of $\mathcal{H}$ on $\mathcal{A}$ is called covariant if the conditions hold*

1. *$\triangleright$ commutes with the product map m (i.e.$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \quad \forall h \in \mathcal{H}, \quad \forall a, b \in \mathcal{A}$),*
2. *$\triangleright$ commutes with the unit map η (i.e. $h \triangleright \mathbb{I}_{\mathcal{A}} = \epsilon(h)\mathbb{I}_{\mathcal{A}}$, $\forall h \in \mathcal{H}$).*

We observe that a covariant action involves the coalgebric sector of the Hopf algebra. Moreover a covariant action preserves both the algebraic and the coalgebraic structure of the Hopf algebra. Two examples of covariant actions are:

[1]Here we only consider left actions.

the canonical action

$$a \overset{can}{\triangleright} b := b_{(1)} \left\langle a, b_{(2)} \right\rangle, \tag{A.18}$$

the adjoint action[2]

$$a \overset{ad}{\triangleright} b := a_{(1)} b S(a_{(2)}). \tag{A.19}$$

One example of a noncovariant action is

$$a \triangleright b := ab, \tag{A.20}$$

that does not preserve the coalgebraic structure of the Hopf algebra.

This completes the review of the definitions and of the properties regarding Hopf algebras which have been used in this thesis.

[2]The adjoint action is defined in the case $\mathcal{H}{=}\mathcal{A}$.

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